كلية العلوم

القسم : **الرياضيات**

السنة : **الثانية**

المادة : معادلات تفاضلية2

المحاضرة : الثالثة/عملي/

{{ A to Z مكتبة }}

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① $x^2y''+x(3+x)y'+(1+x)y=0$ ، $x=0$

نقسم على امثال $y''$

$$y''+\left(\frac{3}{x}+1\right)y'+\left(\frac{1}{x}+1\right)y=0$$

$x=0$ نقطة شاذة للمعادلة لأنها شاذة لـ $p$ و $q$

$x=0$ شاذة نظامية:

$$y=\sum_{n=0}^{\infty}C_n x^{n+\alpha} \quad , \quad y'=\sum_{n=0}^{\infty}(n+\alpha)C_n x^{n+\alpha-1} \quad , \quad y''=\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)C_n x^{n+\alpha-2}$$

نعوض في المعادلة:

$$x^2\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)C_n x^{n+\alpha-2}+(3x+x^2)\sum_{n=0}^{\infty}(n+\alpha)C_n x^{n+\alpha-1}$$
$$+(1+x)\sum_{n=0}^{\infty}C_n x^{n+\alpha}=0$$

$$=\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)C_n x^{n+\alpha}+\sum_{n=0}^{\infty}3(n+\alpha)C_n x^{n+\alpha}+\sum_{n=0}^{\infty}(n+\alpha)C_n x^{n+\alpha+1}$$
$$+\sum_{n=0}^{\infty}C_n x^{n+\alpha}+\sum_{n=0}^{\infty}C_n x^{n+\alpha+1}=0$$

$$=\sum_{n=0}^{\infty}\left[(n+\alpha)(n+\alpha-1)+3(n+\alpha)+1\right]C_n x^{n+\alpha}+\sum_{n=1}^{\infty}\left[(n-1+\alpha)+1\right]C_{n-1}x^{n+\alpha}$$

بفرض $C_0\neq 0 \Leftarrow$

$$(\alpha+1)^2=0 \Rightarrow \alpha=-1$$

$$\sum_{n=0}^{\infty} C_n (n+\alpha+1)^2 x^{n+\alpha} + \sum_{n=0}^{\infty} C_{n-1}(n+\alpha)x^{n+\alpha} = 0$$

$$C_n(n+\alpha+1)^2 + C_{n-1}(n+\alpha) = 0.$$

$$C_n = -\frac{n+\alpha}{(n+\alpha+1)^2} C_{n-1} \; ; \; n \geq 1.$$

$\Leftarrow n=1$

$$C_1 = -\frac{1+\alpha}{(2+\alpha)^2} C_0$$

$$C_2 = -\frac{2+\alpha}{(\alpha+3)^2} C_1 \quad \Leftarrow n=2$$

$$C_3 = -\frac{3+\alpha}{(\alpha+4)^2} C_2 \quad \Leftarrow n=3$$

$$\Rightarrow C_3 = -\frac{3+\alpha}{(\alpha+4)^2} C_2 = -\frac{1+\alpha}{(\alpha+2)(\alpha+3)(\alpha+4)^2}$$

$$\bar{y} = x^{\alpha} \sum_{n=0}^{\infty} C_n x^n = x^{\alpha}\left(1 + \frac{1+\alpha}{(\alpha+2)^2} x + \frac{1+\alpha}{(\alpha+2)(\alpha+3)^2} x^2 + \cdots\right)$$

$$y_1 = \bar{y}\Big|_{\alpha=-1} = x^{-1} \times 1 = \frac{1}{x}$$

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y^2} dx \, . \quad e^{-\int P(x)dx} = e^{-\int \frac{3}{x}+1\, dx} = e^{-3\ln x - x}.$$

$$y_2 = \frac{1}{x} \int \frac{e^{-x}/x^3}{1/x^2} dx = \frac{1}{x} \int \frac{e^{-x}}{x} dx$$

$$y_2 = \frac{1}{x} \int \frac{1}{x} \left[1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} \cdots \right] dx$$

$$= \frac{1}{x}\int\left[\frac{1}{x} - 1 + \frac{x}{2!} - \frac{x^2}{3!} + \dots\right]dx = \frac{1}{x}\left[\ln x - x + \frac{x^2}{4} + \frac{x^3}{18} + \dots\right]$$

$$y = Ay_1 + By_2$$

② $xy'' - (4+x)y' + 2y = 0$ ، $x = 0$

نقسم على امثال $y''$ :

$$y'' - \left(\frac{4}{x} + 1\right)y' + \frac{2}{x}y = 0$$

$x=0$ نقطة شاذة للمعادلة لأن شاذة لـ $p$ و $q$ ← نقطة شاذة نظامية.

$$y = \sum_{n=0}^{\infty} C_n x^{n+\alpha} \quad , \quad y' = \sum_{n=0}^{\infty}(n+\alpha)C_n x^{n+\alpha-1}$$

$$y'' = \sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)C_n x^{n+\alpha-2}$$

نعوض

$$\sum_{n=0}^{\infty}(n+\alpha-1)(n+\alpha)C_n x^{n+\alpha-1} - 4\sum_{n=0}^{\infty}(n+\alpha)C_n x^{n+\alpha-1} - \sum_{n=0}^{\infty}(n+\alpha)C_n x^{n+\alpha} + 2\sum_{n=0}^{\infty}C_n x^{n+\alpha} = 0$$

$$\sum_{n=0}^{\infty}((n+\alpha-1-4)(n+\alpha))C_n x^{n+\alpha-1} - \sum_{n=0}^{\infty}(n+\alpha-2)C_n x^{n+\alpha}$$

$$(\alpha-5)(\alpha)C_0 x^{\alpha-1} + \sum_{n=1}^{\infty}(n+\alpha-5)(n+\alpha)C_n x^{n+\alpha-1} - \sum_{n=1}^{\infty}(n+\alpha-3)C_{n-1}x^{n+\alpha-1} = 0$$

$\Leftarrow C_0 \neq 0$

$\alpha(\alpha-5) = 0 \Rightarrow$ إما $\alpha_1 = 0$ أو $\alpha_2 = 5$.

$$\sum_{n=1}^{\infty}(n+\alpha-5)(n+\alpha)C_n\, x^{n+\alpha}-\sum_{n=1}^{\infty}(n+\alpha-3)C_{n-1}\, x^{n+\alpha-1}=0$$

$$(n+\alpha-5)(n+\alpha)C_n-(n+\alpha-3)C_{n-1}=0$$

$$C_n=\frac{(n+\alpha-3)}{(n+\alpha-5)(n+\alpha)}C_{n-1}$$

نفرض $\alpha=0$

$$C_n=\frac{n-3}{n(n-5)}C_{n-1}$$

من اجل $n=5$ ، $C_5$ غير معينة.

$$C_1=\frac{\alpha-2}{(\alpha-4)(\alpha+1)}C_0$$

$$C_2=\frac{\alpha-1}{(\alpha-3)(\alpha+2)}C_1=\frac{(\alpha-1)(\alpha-2)}{(\alpha-3)(\alpha+2)(\alpha-4)(\alpha+1)}$$

$$\bar{y}=x^{\alpha}\left(C_0+\left(\frac{\alpha-2}{(\alpha-4)(\alpha+1)}C_0\right)x+\frac{(\alpha-1)(\alpha-2)}{(\alpha-3)(\alpha+2)(\alpha-4)(\alpha+1)}C_0x^3+\cdots\right)$$

$$\frac{\partial\bar{y}}{\partial\alpha}=x^{\alpha}\ln x\left(C_0+\frac{\alpha-2}{(\alpha-4)(\alpha+1)}C_0x+\frac{(\alpha-1)(\alpha-2)}{(\alpha-3)(\alpha+2)(\alpha-4)(\alpha+1)}C_0x^2\right)$$

$$+x^{\alpha}\left(\frac{-\alpha^2+4\alpha-10}{(\alpha-4)^2(\alpha+1)^2}x+\cdots\right)$$

نفرض $C_0=\alpha-\alpha_0$ ⟸ $C_0=\alpha$

$$\bar{y}=x^{\alpha}\left(\alpha+\frac{\alpha-2}{(\alpha-4)(\alpha+1)}\alpha x+\cdots\right)$$

$$\frac{\partial\bar{y}}{\partial\alpha}=x^{\alpha}\ln\left[\alpha+\frac{\alpha(\alpha-2)}{(\alpha-4)(\alpha+1)}x+\cdots\right]+x^{\alpha}\left[\frac{-\alpha^2+4\alpha-10}{(\alpha-4)^2(\alpha+1)^2}x+\cdots\right]$$

$$y_1=\bar{y}\Big|_{\alpha=0}\ ,\ y_2=\frac{\partial y}{\partial\alpha}\Big|_{\alpha=0}\Rightarrow y=Ay_1+By_2$$

③ $xy'' - (x+1)y' - y = 0$

$$y = \sum_{n=0}^{\infty} C_n x^{n+\alpha}, \quad y' = \sum_{n=0}^{\infty} (n+\alpha) C_n x^{n+\alpha-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+\alpha-1)(n+\alpha) C_n x^{n+\alpha-2}$$

نعوض

$$\sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1) C_n x^{n+\alpha-1} - \sum_{n=0}^{\infty} (n+\alpha) C_n x^{n+\alpha} - \sum_{n=0}^{\infty} (n+\alpha) C_n x^{n+\alpha-1}$$

$$- \sum_{n=0}^{\infty} C_n x^{n+\alpha} = 0$$

$$= \sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-2) C_n x^{n+\alpha-1} - \sum_{n=0}^{\infty} (n+\alpha) C_{n-1} x^{n+\alpha-1} = 0$$

$n=0 \Rightarrow \alpha(\alpha-2) = 0 \Rightarrow \alpha_1 = 0$ أو $\alpha_2 = 2$

$$C_n = \frac{1}{n+\alpha-2} C_{n-1}$$

$$\alpha = 0 \Rightarrow C_n = \frac{1}{n-2} C_{n-1},$$

$C_2$ غير معينة و $C_0 = \alpha \Leftarrow$

$$C_1 = \frac{1}{\alpha-1} C_0, \quad C_0 = \frac{\alpha}{\alpha-1}$$

$$C_2 = \frac{1}{\alpha} C_1 = \frac{1}{\alpha(\alpha-1)}(\alpha) = \frac{1}{\alpha-1}$$

$$C_3 = \frac{1}{\alpha+1} C_2 = \frac{1}{(\alpha+1)(\alpha-1)}$$

$$\bar{y} = x^{\alpha}\left[\alpha + \frac{\alpha}{\alpha-1}x + \frac{1}{\alpha-1}x^2 + \frac{1}{\alpha^2-1}x^3 + \cdots\right]$$

$$y_1 = \bar{y}\Big|_{x=0}$$

$$y_1 = 1[1 - x^2 - x^3 \dots]$$

$$\frac{\partial \bar{y}}{\partial x} = x^x \ln\left[x + \frac{x}{x-1}x + \dots\right] + x^x\left[1 - \frac{1}{(x-1)^2}x - \frac{1}{(x-1)^2}x^2 + \frac{2x}{(x^2-1)^2}x^3 + \dots\right]$$

$$y_2 = \frac{\partial \bar{y}}{\partial x}\Big|_{x=0} = \ln x\left[-x^2 - x^3 + \dots\right] + \left[1 - x - x^2 + \frac{1}{4}x^4 + \dots\right]$$

$$y = Ay_1 + By_2$$

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$$4x^3y'' + 6x^2y + y = 0$$

نضع $x = \frac{1}{t} \Leftarrow y'_x = -y'_t t^2$ :

$$y''_x = t^4 \cdot y''_t + 2t^3 y'_t$$

$$4(ty'' + 2y'_t) - 6y'_t + y = 0$$

$$4ty'' + 2y'_t + y = 0$$

$t=0$ نقطة شاذة نظامية

محلولة بالتفريق:

بالنهاية: $y = A\left[1 - \frac{1}{2}t + \dots\right] + Bt^{\frac{1}{2}}\left[1 - \frac{1}{6} + \dots\right]$

$$A = \left[1 - \frac{1}{2}x^{-1} + \dots\right] + Bx^{-\frac{1}{2}}\left[1 - \frac{1}{6}x^{-1} + \dots\right]$$

انتهت المحاضرة