كلية العلوم

القسم : **الفيزياء**

السنة : **الاولى**

المادة : تحليل رياضي2

المحاضرة : الثانية/نظري/

{{ A to Z مكتبة }}

Facebook Group : A to Zمكتبة

الطرق الأساسية في المكاملة:

1) طريقة تغيير المتحول: تعتمد على إدخال متحول جديد t

$$\int f(x)\,dx = \int f[\varphi(t)]\,\varphi'(t)\,dt$$

مثال: أوجد: $I = \int \sin^3 x \cdot \cos x\, dx$

نفرض: $t = \sin x \Rightarrow dt = \cos x\, dx$

$$I = \int t^3 dt = \frac{t^4}{4} + c$$

مثال: $I = \int (x^4 + 3x)^7 (4x^3 + 3)\, dx$

نفرض: $t = x^4 + 3x \Rightarrow dt = (4x^3 + 3)\, dx$

$$I = \int t^7 dt = \frac{1}{8} t^8 + c = \frac{1}{8}(x^4 + 3x)^8 + c$$

مثال: $I = \int \sin^3 x\, dx$

$= \int \sin^2 x \sin x\, dx$

تذكر: $\boxed{\sin^2 x = 1 - \cos^2 x}$

$= \int (1 - \cos^2 x) \sin x\, dx$

نفرض: $t = \cos x \Rightarrow dt = -\sin x\, dx$

$$I = \int (1 - t^2)(-dt) = \int (t^2 - 1)\, dt$$

$$= \frac{t^3}{3} - t + c = \frac{1}{3}\cos^3 x - \cos x + c$$

التكاملات من النمط:

$$\int \frac{d\varphi(t)}{\varphi(t)} = \ln|\varphi(t)|$$

مثال:

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$$

$$t = \cos x \Rightarrow dt = -\sin x \, dx$$

$$\int \tan x \, dx = \int -\frac{dt}{t} = -\ln|t| = -\ln|\cos x|$$

مثال:

$$I = \int \frac{dx}{(x^2+2x+2)\,\mathrm{arctg}(x+1)}$$

$$t = \mathrm{arctg}(x+1) \Rightarrow dt = \frac{1}{1+(x+1)^2} dx = \frac{1}{x^2+2x+2} dx$$

$$I = \int \frac{1}{\mathrm{arctg}(x+1)} \frac{dx}{x^2+2x+2}$$

$$= \int \frac{1}{t} dt = \ln|t| + c = \ln|\mathrm{arctg}(x+1)| + c$$

مثال:

$$I = \int \frac{x\,dx}{\sqrt{1-x^2}}$$

$$t = 1-x^2 \Rightarrow dt = -2x\,dx \Rightarrow x\,dx = -\frac{1}{2} dt$$

$$I = \int \frac{-\frac{1}{2}dt}{\sqrt{t}} = -\frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\frac{1}{2} \cdot 2\sqrt{t} + c$$

$$= -\sqrt{t} + c = -\sqrt{1-x^2} + c$$

مثال:

$$I = \int \frac{e^x dx}{e^{2x}+1}$$

$$t = e^x \Rightarrow dt = e^x dx$$

$$I = \int \frac{dt}{t^2+1} = \mathrm{arctg}\, t + c = \mathrm{arctg}(e^x) + c$$

مثال: $I = \int \frac{\cos x}{1+\sin^2 x} \, dx$

نفرض $t = \sin x \Rightarrow dt = \cos x \, dx$

$I = \int \frac{dt}{1+t^2} = \operatorname{arctg} t + c = \operatorname{arctg}(\sin x) + c$

التكاملات من النمط: $I = \int g(x^2) \, x \, dx$

في هذه الحالة نفرض: $x^2 = t \Rightarrow dt = 2x \, dx$

$I = \int g(t) \frac{dt}{2}$

مثال: $I = \int e^{x^2} x \, dx$

$t = x^2 \Rightarrow dt = 2x \, dx$

$I = \int e^t \frac{dt}{2} = \frac{1}{2} e^t + c = \frac{1}{2} e^{x^2} + c$

مثال: $I = \int \frac{x \, dx}{\sqrt{1-x^4}}$ ، نفرض: $t = x^2$

$\Rightarrow dt = 2x \, dx$

$= \int \frac{\frac{1}{2} dt}{\sqrt{1-t^2}} = \frac{1}{2} \arcsin t + c = \frac{1}{2} \arcsin x^2 + c$

مثال: $I = \int \frac{x}{\cos^2(x^2)} dx$ ، نفرض: $t = x^2$

$\Rightarrow dt = 2x \, dx$

$= \int \frac{\frac{1}{2} dt}{\cos^2(t)} = \frac{1}{2} \tan t + c = \frac{1}{2} \tan(x^2) + c$

التكاملات من النمط: $I = \int g(\ln x) \frac{dx}{x}$

في هذه الحالة نفرض: $t = \ln x$

$dt = \frac{dx}{x} \Rightarrow I = \int g(t) \, dt$

مثال: نفرض: $t = \ln x$, $I = \int \frac{\ln x}{x}\, dx$

$\Rightarrow dt = \frac{dx}{x}$

$I = \int t\, dt = \frac{1}{2} t^2 + c = \frac{1}{2} (\ln x)^2 + c$

مثال: نفرض: $t = \ln x$, $I = \int \frac{dx}{x \cdot \ln x}$

$\Rightarrow dt = \frac{dx}{x}$

$= \int \frac{1}{t}\, dt = \ln|t| + c = \ln|\ln x| + c$

مثال: نفرض: $t = \ln x$, $I = \int \frac{dx}{(\ln x)^2 \cdot x}$

$\Rightarrow dt = \frac{dx}{x}$

$= \int \frac{dt}{t^2} = \int t^{-2}\, dt = \frac{t^{-1}}{-1} + c = \frac{-1}{t} + c = \frac{-1}{\ln x} + c$

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2) التكامل بالتجزئة:

$d(u.v) = v\, du + u\, dv$

$u\, dv = d(u.v) - v.du$

قانون التكامل بطريقة التجزئة: $\Rightarrow \int u\, dv = u v - \int v\, du$

المجموعة الأولى:

كثير حدود

$\int P_n(x)\, e^{kx}\, dx$

$\int P_n(x) \cos kx\, dx$

$\int P_n(x) \sin kx\, dx$

$\Rightarrow$ في هذه الحالة نعوض: $u = P_n(x)$

dv : $e^{kx}\, dx$, $\cos kx\, dx$, $\sin kx\, dx$

مثال: أوجد: $I = \int x^2 \sin x\, dx$

$u = x^2 \Rightarrow du = 2x\,dx$

$dv = \sin x\,dx \Rightarrow v = -\cos x$

$I = -x^2 \cos x - \int (-\cos x)(2x\,dx)$

$I = -x^2 \cos x + 2\underbrace{\int x \cos x\,dx}_{I_1} + C$

$I_1 = \int x \cos x\,dx$

$u = x \Rightarrow du = dx \quad , dv = \cos x\,dx \Rightarrow v = \sin x$

$I_1 = x \sin x - \int \sin x\,dx = x \sin x + \cos x + C_1$

$I = -x^2 \cos x + 2x \sin x + 2 \cos x + C_2$ $\quad C + C_1$

مثال:

$I = \int (x^2 - 6x + 5)\,e^{3x}\,dx$

$u = x^2 - 6x + 5 \Rightarrow du = (2x - 6)\,dx$

$dv = e^{3x}\,dx \Rightarrow v = \frac{1}{3} e^{3x}$

$I = \frac{1}{3}(x^2 - 6x + 5)e^{3x} - \int \frac{1}{3}\underbrace{(2x-6)e^{3x}\,dx}_{I_1}$

$I_1 : u = 2x - 6 \Rightarrow du = 2\,dx$

$dv = e^{3x}\,dx \Rightarrow v = \frac{1}{3} e^{3x}$

$I_1 = \frac{1}{3}(2x-6)e^{3x} - \int \frac{2}{3} e^{3x}\,dx$

$= \frac{1}{3}(2x-6)e^{3x} - \frac{2}{9} e^{3x} + C$

$I = \frac{1}{3}(x^2 - 6x + 5)e^{3x} - \frac{1}{9}(2x-6)e^{3x} + \frac{2}{27} e^{3x} + C$

المجموعة الثانية:

$\int P_n(x)\,\ln x\,dx$

$\int P_n(x)\,\arccos x\,dx$

$\int P_n(x)\ \arcsin x\, dx$

$\int P_n(x)\ \arctan x\, dx$

$\int P_n(x)\ \text{arc}\cot x\, dx$

$\Rightarrow dv = P_n(x)\, dx \quad , \quad u = \begin{cases} \ln x \\ \text{arc} \end{cases}$

مثال:

$I = \int x^2 \ln x \, dx$

$u = \ln x \Rightarrow du = \frac{dx}{x}$

$dv = x^2 dx \Rightarrow v = \frac{x^3}{3}$

$I = \frac{x^3}{3}\ln x - \int \frac{dx}{x}\,\frac{x^3}{3} = \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\,dx$

$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$

مثال:

$I = \int x \ln(x^2)\, dx$

$t = x^2 \Rightarrow dt = 2x\, dx$

$I = \int \ln t \,\frac{dt}{2} \Rightarrow I = \frac{1}{2}\int \ln t\, dt$

$u = \ln t \Rightarrow du = \frac{dt}{t} \quad , dv = dt \Rightarrow v = t$

$I = t \ln t - \int \frac{dt}{t}\, t = t \ln t - \int dt$

$= t \ln t - t \Rightarrow I = x^2 \ln(x^2) - x^2$

مثال:

$I = \int \ln\sqrt{1-x}\, dx$

$u = \ln\sqrt{1-x} \Rightarrow du = \frac{\frac{-1}{2\sqrt{1-x}}}{\sqrt{1-x}}\, dx = \frac{-1}{2(1-x)}\, dx$

$dv = dx \Rightarrow v = x$

$I = x\ln(\sqrt{1-x}) - \int \frac{-x}{2(1-x)}\, dx$

$I = x\ln(\sqrt{1-x}) + \frac{1}{2}\int \frac{x}{1-x}\, dx$

$-x+1 \,\overline{)\, x \,}\ \ -1$

$\underline{x - 1}$

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$I = x\ln(\sqrt{1-x}) + \frac{1}{2}\int\left[-1 + \frac{1}{1-x}\right]dx$

$= x\ln(\sqrt{1-x}) + \frac{1}{2}\left[-x - \ln|1-x|\right] + c$

مثال: $I = \int \arcsin x\, dx$

$u = \arcsin x \Rightarrow du = \frac{dx}{\sqrt{1-x^2}}$, $dv = dx \Rightarrow v = x$

$I = x \arcsin x - \int \frac{x\, dx}{\sqrt{1-x^2}}$

لحساب: $J = \int \frac{x\,dx}{\sqrt{1-x^2}}$ نفرض: $t = x^2 \Rightarrow dt = 2x\,dx$

$J = \int \frac{\frac{1}{2}dt}{\sqrt{1-t}} = \frac{1}{2}(-2)\sqrt{1-t}$

$= -\sqrt{1-t} = -\sqrt{1-x^2} \Rightarrow I = x\arcsin x + \sqrt{1-x^2} + c$

المجموعة الثالثة:

1) $\int e^{ax}\sin bx\, dx$

$\int e^{ax}\cos bx\, dx$

نفرض: $u = \sin bx$ أو $\cos bx$, أو $u = e^{ax}$

$dv =$ الباقي من التكامل

2) $\int \sin(\ln x)\,dx$, $\int \cos(\ln x)\,dx$

نفرض: $u = \sin(\ln x)$ أو $\cos(\ln x)$, $dv = dx$

مثال: $I = \int e^{-x}\cos\frac{x}{2}\,dx$

$u = e^{-x} \Rightarrow du = -e^{-x}\,dx$

$dv = \cos\frac{x}{2}\,dx \Rightarrow v = 2\sin\frac{x}{2}$

$I = 2e^{-x}\sin\frac{x}{2} + 2\underbrace{\int e^{-x}\sin\frac{x}{2}\,dx}_{J}$

$u = e^{-x} \Rightarrow du = -e^{-x}\,dx$

$dv = \sin\frac{x}{2}\,dx \Rightarrow v = -2\cos\frac{x}{2}$

$J = -2e^{-x}\cos\frac{x}{2} - 2\int e^{-x}\cos\frac{x}{2}\,dx$

$J = -2e^{-x}\cos\frac{x}{2} - 2I$

$I = 2e^{-x}\sin\frac{x}{2} - 4e^{-x}\cos\frac{x}{2} - 4I$

$5I = 2e^{-x}\sin\frac{x}{2} - 4e^{-x}\cos\frac{x}{2}$

$I = \frac{2}{5}e^{-x}\sin\frac{x}{2} - \frac{4}{5}e^{-x}\cos\frac{x}{2}$

$I = \int \cos(\ln x)\,dx$ مثال: $\boxed{(\cos u)' = -\sin u \times u'}$

$u = \cos(\ln x) \Rightarrow du = -\sin(\ln x)\,\frac{1}{x}\,dx$

$dv = dx \Rightarrow v = x \Rightarrow I = x\cos(\ln x) + \underbrace{\int \sin(\ln x)\,dx}_{J}$

$J: u = \sin(\ln x) \Rightarrow du = \cos(\ln x)\,\frac{1}{x}\,dx \quad , dv = dx \Rightarrow v = x$

$J = x\sin(\ln x) - \int \cos(\ln x)\,dx$

$J = x\sin(\ln x) - I$

$I = x\cos(\ln x) + x\sin(\ln x) - I$

$2I = x\cos(\ln x) + x\sin(\ln x) + c$

$I = \frac{1}{2}[x\cos(\ln x) + x\sin(\ln x) + c]$

انتهت المحاضرة