كلية العلوم

القسم : **الرياضيات**

السنة : **الثالثة**

المادة : معادلات تفاضلية جزئية

المحاضرة : السابعة/عملي/

{{ مكتبة A to Z }}

Facebook Group : مكتبة A to Z

القسم: رياضيات

السنة: الثالثة

المادة: معادلات تفاضلية جزئية

المحاضرة: 7 عملي

حل المثال السؤال الأول:

$$U_{xx} + U_{yy} = 0$$

$$U(x,0) = 0$$

$$U(x,3) = \frac{x}{2}$$

$$U(0,y) = \sin \frac{4\pi}{3} y$$

$$U(2,y) = 7$$

الحل: المسألة الأولى:

$$U(x,3) = \frac{x}{2}$$

$$U(x,0) = U(0,y) = U(2,y) = 0$$

$$U(x,y) = XY$$

$$U_{xx} = X''Y$$

$$U_{yy} = Y''X$$

$$\Rightarrow X''Y + Y''X = 0$$

$$\frac{X''}{X} = \frac{-Y''}{Y} = \lambda$$

X تابع لـ x فقط — Y تابع لـ y فقط

$$U(x,0) = X(x)\,Y(0) = 0 \Rightarrow Y(0) = 0$$

$$U(0,y) = X(0)\,Y(y) = 0 \Rightarrow X(0) = 0$$

$$U(2,y) = X(2)\,Y(y) = 0 \Rightarrow X(2) = 0$$

$X'' - \lambda X = 0$

$X = Ax + B \Leftarrow \lambda = 0$ (1)

$X(0) = 0 \Rightarrow A(0) + B = 0 \Rightarrow \boxed{B = 0}$

$X(2) = 0 \Rightarrow \boxed{A = 0} \Rightarrow X(x) = 0$

مرفوض

$\lambda > 0$ (2)

$X(x) = A e^{\sqrt{\lambda}\, x} + B e^{-\sqrt{\lambda}\, x}$

$X(0) = A + B = 0 \Rightarrow A = -B$

$X(2) = A\left(e^{2\sqrt{\lambda}} - e^{-2\sqrt{\lambda}}\right) = 0 \Rightarrow A = B = 0$

$\Rightarrow X(x) = 0$ مرفوض

$\lambda < 0$ (3)

$X(x) = A \cos \sqrt{\lambda}\, x + B \sin \sqrt{\lambda}\, x$

$X(0) = A = 0$

$X(2) = \underbrace{B}_{\neq 0} \sin 2\sqrt{\lambda} = 0 \Rightarrow \sin 2\sqrt{\lambda} = 0$

$\Rightarrow 2\sqrt{\lambda} = n\pi$

$\Rightarrow \lambda = -\frac{n^2\pi^2}{4}$

$\Rightarrow \boxed{X_n(x) = B_n \sin \frac{n\pi}{2} x}$

$y'' + \lambda y = 0$

$\Leftarrow -\lambda > 0 \quad \Leftarrow \lambda < 0$

$y(y) = c e^{\sqrt{\lambda}\, y} + D e^{-\sqrt{\lambda}\, y}$

$y(y) = c_1 \operatorname{ch} \sqrt{\lambda}\, y + c_2 \operatorname{sh} \sqrt{\lambda}\, y$

$$Y(0) = C_1 = 0$$

$$\Rightarrow \boxed{Y(y) = C_2 \sinh \frac{n\pi}{2} y}$$

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi}{2} x \cdot \sinh \frac{n\pi}{2} y$$

$$u(x,3) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi}{2} x \cdot \sinh \frac{3n\pi}{2} = \frac{x}{2}$$

$$C_n \sinh \frac{3n\pi}{2} = \frac{2}{\ell} \int \frac{x}{2} \sin \frac{n\pi}{2} x \, dx$$

$$= \int_0^2 \frac{x}{2} \sin \frac{n\pi}{2} x \, dx$$

$$u = \frac{x}{2} \quad , \quad dv = \sin \frac{n\pi}{2} x \, dx$$

$$du = \frac{dx}{2} \quad , \quad v = \frac{-2}{n\pi} \cos \frac{n\pi}{2} x$$

$$\Rightarrow C_n \sinh \frac{3n\pi}{2} = \left[\frac{-x}{n\pi} \cos \frac{n\pi}{2} x \right] - \frac{1}{2} \int \cos \frac{n\pi}{2} x$$

$$= \left(\frac{-2}{n\pi} \right) (-1)^n - 0 - \frac{1}{2} \left[\frac{2}{n\pi} \sin \frac{n\pi}{2} x \right]_0^2$$

$$\Rightarrow \left(\frac{-2}{n\pi} \right) (-1)^n$$

$$\Rightarrow C_n = \left(\frac{-2}{n\pi} \right) (-1)^n \left(\sinh \frac{3n}{2} (n) \right)^{-1} \Rightarrow u = (x,y) = \sum_{n=1}^{\infty} (-1)^n \frac{-2}{n\pi}$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{-2}{n\pi} \left(\sinh \frac{3n(n)}{2} \right)^{-1} \cdot \sin \frac{n(n)}{2} x \sinh \frac{n(n)}{2} y$$

$u(0,y) = \sin\frac{4\pi}{3}y$: الشرط الحدي

$u(x,0) = u(x,3) = u(2,y) = 0$

$u(x,y) = XY$

$u_{xx} = X''Y$

$u_{yy} = Y''X$

$\Rightarrow X''Y + Y''X = 0$

$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$

تابع لـ y فقط تابع لـ x فقط

$u(x,0) = X(x)\,Y(0) = 0 \Rightarrow \boxed{Y(0)=0}$

$u(x,3) = X(x)\,Y(3) = 0 \Rightarrow \boxed{Y(3)=0}$

$u(2,y) = X(2)\,Y(y) = 0 \Rightarrow \boxed{X(2)=0}$

$Y'' + \lambda Y = 0$

$\lambda = 0$ ①

$m^2 + \lambda = 0$

$Y = Ay + B$

$Y(0) = B = 0$

$Y(3) = 3A = 0 \Rightarrow A = 0$

مرفوض

$\lambda < 0$ ②

$Y = Ae^{\sqrt{-\lambda}\,y} + Be^{\sqrt{-\lambda}\,y}$

$\left.\begin{matrix} Y(0)=0 \\ Y(3)=0 \end{matrix}\right\} \Rightarrow A = B = 0$

الحل مرفوض

$\Leftarrow \lambda > 0$ ③

$$Y(y) = A\cos\sqrt{\lambda}\,y + B\sin\sqrt{\lambda}\,y$$

$Y(0) = A + B = 0 \Rightarrow A = -B$ $\quad Y(0) = A = 0$

$$Y(3) = B\sin 3\sqrt{\lambda} = 0 \Rightarrow \sin 3\sqrt{\lambda} = 0$$

$$\Rightarrow 3\sqrt{\lambda} = n\pi$$

$$\Rightarrow \sqrt{\lambda} = \frac{n\pi}{3} \Rightarrow \lambda = \frac{n^2\pi^2}{9}$$

$$\boxed{Y_n(y) = B_n \sin\frac{n\pi}{3}y}$$

$$X(x) = C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x}$$

نلاحظ أن التابع الأسي لا يفيدنا في حساب الثوابت

لذلك سنستخدم التوابع القطعية:

$$X(x) = M\,ch\sqrt{\lambda}\,x + N\,sh\sqrt{\lambda}\,x$$

نلاحظ بأن هذه التوابع تفيدنا في حساب الثوابت

أيضاً لذلك سنستخدم دساتير تجوع الزوايا.

$$X(x) = q\,ch\frac{n\pi}{3}(x-2) + p\,sh\frac{n\pi}{3}(x-2)$$

$$X(2) = \boxed{q = 0}$$

$$X_n(x) = p_n\,sh\frac{n\pi}{3}(x-2)$$

$$U_2(x,y) = \sum_{n=1}^{\infty} D_n\,sh\frac{n\pi}{3}(x-2)\sin\frac{n\pi}{3}y$$

$$U(0,y) = \sum_{n=1}^{\infty} D_n\,sh\frac{-2n\pi}{3}\sin\frac{n\pi}{3}y = \sin\frac{4\pi}{3}y$$

بالمطابقة نجد أن:

$$D_4 \left(\sinh \frac{-8\pi}{3}\right) = 1 \Rightarrow D_4 = \left(\sinh \frac{-8\pi}{3}\right)^{-1}$$

$$\boxed{u_2(x,y) = \left(\sinh \frac{-8}{3}\pi\right)^{-1} \sinh \frac{n\pi}{3} x \sin \frac{n\pi}{3} y}$$

الـــألة الثالثة :

$$u_{xx} + u_{yy} = 0$$

$$u(2,y) = 7$$

$$u(x,0) = u(x,3) = u(0,y) = 0$$

نفرض أن المعادلة حلها بالشكل :

$$u(x,y) = X(x)\, Y(y)$$

$$u_{xx}(x,y) = X''(x)\, Y(y)$$

$$u_{yy}(x,y) = X(x)\, Y''(y)$$

نعوض في المعادلة نجد أن :

$$X''(x)\, Y(y) + X(x)\, Y''(y) = 0$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda$$

تابع x فقط — تابع y فقط

فنحل كل المعادلتين :

$$X''(x) - \lambda X(x) = 0 \quad \cdots (1)$$

$$Y''(y) + \lambda Y(y) = 0 \quad \cdots (2)$$

ولدينا الشروط :

$$u(x,0) = X(x)\, Y(0) = 0 \Rightarrow \boxed{Y(0) = 0}$$

($X(x) \neq 0$)

$$u(x,3) = X(x)\, Y(3) = 0 \Rightarrow \boxed{Y(3) = 0}$$

($X(x) \neq 0$)

$u(0,y) = X(0)\underbrace{Y(y)}_{\neq 0} = 0 \Rightarrow \boxed{X(0)=0}$

نبدأ بحل المعادلة:

$$Y''(y) + \lambda Y(y) = 0$$

$$m^2 + \lambda = 0 \Rightarrow m = \pm\sqrt{-\lambda}$$

نناقش حسب قيم λ:

[1] $\lambda = 0 \Rightarrow$

$Y(y) = A + By$

$Y(0) = \boxed{A=0}$

$Y(3) = 3B = 0 \Rightarrow B = 0$

حل صفري مرفوض. $Y(y) = 0$

[2] $\lambda < 0 \Rightarrow$

$Y(y) = Ae^{+\sqrt{-\lambda}\,y} + Be^{\sqrt{-\lambda}\,y}$

$Y(0) = A + B = 0 \Rightarrow \boxed{A = -B}$

$Y(3) = A(\underbrace{e^{3\sqrt{-\lambda}} - e^{-3\sqrt{-\lambda}}}_{\neq 0}) = 0 \Rightarrow \boxed{A=0}$

$\Rightarrow B = 0 \Rightarrow$ حل صفري مرفوض $Y(y)$

[3] $\lambda > 0 \Rightarrow$

$Y(y) = A\cos\sqrt{\lambda}\,y + B\sin\sqrt{\lambda}\,y$

$Y(0) = \boxed{A=0}$

$Y(3) = B\sin 3\sqrt{\lambda} = 0 \Rightarrow \sin 3\sqrt{\lambda} = 0$

$3\cdot\sqrt{\lambda} = n\pi \Rightarrow \sqrt{\lambda} = \frac{n\pi}{3} \Rightarrow \lambda = \frac{n^2\pi^2}{9}$

$$Y_n(y) = B_n \sin\frac{n\pi}{3}y$$

$$X(x) = C_1 e^{\sqrt{\lambda} x} + C_2 e^{-\sqrt{\lambda} x}$$

نلاحظ أن التابع الأساسي X يغيرنا في حالة
حتى الثوابت لذلك نستخدم التوابع الزائدية:

$$X(x) = M \operatorname{ch}\sqrt{\lambda} x + N \operatorname{sh}\sqrt{\lambda} x$$

$$X(0) = \boxed{M = 0}$$

$$X_n(x) = N_n \operatorname{sh} \frac{n\pi}{3} x$$

$$u_3(x,y) = \sum_{n=1}^{\infty} D_n \operatorname{sh} \frac{n\pi}{3} x \sin \frac{n\pi}{3} y$$

$$u(2,y) = \sum_{n=1}^{\infty} D_n \operatorname{sh} \frac{2n\pi}{3} \sin \frac{n\pi}{3} y = 7$$

$$D_n \operatorname{sh} \frac{2n\pi}{3} = \frac{2}{3} \int_0^3 7 \sin \frac{n\pi}{3} y \, dy$$

$$D_n \operatorname{sh} \frac{2n\pi}{3} = \frac{2}{3} \left[-\frac{3 \times 7}{n\pi} \cos \frac{n\pi}{3} \right]_0^3$$

$$= \frac{14}{n\pi} \left[(-1)^{n+1} + 1 \right]$$

$$\Rightarrow D_n = \frac{14}{n\pi} \left[(-1)^{n+1} + 1 \right] \left(\operatorname{sh} \frac{2n\pi}{3} \right)^{-1}$$

$$u_3(x,y) = \sum_{n=1}^{\infty} \frac{14}{n\pi} \left[(-1)^{n+1} + 1 \right] \left(\operatorname{sh} \frac{2n\pi}{3} \right)^{-1} \operatorname{sh} \frac{n\pi}{3} x \sin \frac{n\pi}{3} y$$

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y)$$

السؤال الثاني: أوجد ما يلي:

(1) $\mathcal{L}(t^2 e^t) = \frac{2!}{(s-1)^3} = \frac{2}{(s-1)^3}$

(2) $\mathcal{L}(t e^{2t}) = \frac{1!}{(s-2)^2} = \frac{1}{(s-2)^2}$

(3) $\mathcal{L}(4t^2 - 3\cos 2t + 5e^t) = 4\frac{2!}{s^3} - 3\frac{s}{s^2+4} + 5\cdot\frac{1}{s+1}$

$= \frac{8}{s^3} - \frac{3s}{s^2+4} + \frac{5}{s+1}$

(4) $\mathcal{L}(e^{-4x}\sin x) = \frac{1}{(s+4)^2+1}$

(5) $\mathcal{L}(x^7 e^{4x}) = \frac{7!}{(s-4)^8}$

(6) $\mathcal{L}(x\cos 5x) = (-1)^1 \frac{d}{ds}[\mathcal{L}(\cos 5x)]$

$= -\frac{d}{ds}\left[\frac{s}{s^2+25}\right] = -\left[\frac{s^2+25-2s^2}{(s^2+25)^2}\right]$

$\mathcal{L}(x\cos 5x) = \frac{s^2-25}{(s^2+25)^2}$

(7) $\mathcal{L}(4e^{5t} + 6t^3 - 3\sin 4t + 2\cos 2t) =$

$\frac{4}{s-5} + \frac{6\times 3!}{s^4} - \frac{3\times 4}{s^2+16} + \frac{2s}{s^2+4}$

$$= \frac{4}{s-5} + \frac{36}{s^5} - \frac{12}{s^2+16} + \frac{2s}{s^2+4}$$

السؤال الثالث: أوجد ناتج ما يلي:

① $\mathcal{L}^{-1}\left(\frac{1}{s+3}\right) = e^{-3x}$

② $\mathcal{L}^{-1}\left(\frac{4}{s-2} + \frac{3s}{s^2+16} + \frac{5}{s^2+4}\right) = 4e^{2x} - 3\cos 4x + 5\sin 2x$

③ $\mathcal{L}^{-1}\left(\frac{1}{s^2-2s+5}\right) = \mathcal{L}^{-1}\left(\frac{1}{(s-1)^2+4}\right) = e^{x}\sin 2x$

④ $\mathcal{L}^{-1}\left(\frac{-2s}{(s^2+1)^2}\right) = \mathcal{L}^{-1}\mathcal{L}^{-1}\left(\left(\frac{1}{s^2+1}\right)'\right) = (-1)\,t\sin t = -t\sin t$

يوجد تتمة لهذا السؤال بالصورة المقابلة

انتهت الصورة