كلية العلوم

القسم : **الرياضيات**

السنة : **الثالثة**

المادة : معادلات تفاضلية جزئية

المحاضرة : الخامسة/عملي/

السؤال الأول:

لتكن لدينا معادلة التوصيل الحراري التالية:

$$u_t = u_{xx} + 4$$
$$u_x(0,t) = 5$$
$$u_x(\ell,t) = 6$$
$$u(x,0) = f(x)$$

أوجد الطاقة الحرارية الكلية:

الحل:

أولاً نجد:

$$\int_0^{\ell} u_t(t,x)\,dx = \int_0^{\ell} [u_{xx} + 4]\,dx = [4x + 4x]_0^{\ell}$$
$$= u_x(\ell,t) - u_x(0,t) + 4\ell$$
$$= 6 - 5 + 4\ell = 1 + 4\ell$$

الطاقة الحرارية الكلية:

$$\int (1+4\ell)\,dt = t + 4\ell t + c$$
$$u(x,0) = f(x)$$
$$c = \int_0^{\ell} u(x,0)\,dx = \int_0^{\ell} f(x)\,dx$$

$$E(t) = t + 4\ell t + \int_0^{\ell} f(x)\,dx$$

السؤال الثاني :

لتكن لدينا معادلة التوصيل الحراري التالية :

$u_t = u_{xx} + x$

$u_x(0,t) = \beta$

$u_x(\ell,t) = 7$

$u(x,0) = x+1$

أوجد الطاقة الحرارية الكلية.

الحل :

أولاً نجد :

$$\int_0^{\ell} u_t(t,x)\,dx = \int_0^{\ell} (u_{xx}+x)\,dx = \left[u_x + \frac{x^2}{2}\right]_0^{\ell}$$

$$= u_x(\ell,t) - u_x(0,\ell) + \frac{\ell^2}{2} = 7-\beta+\frac{\ell^2}{2}$$

الطاقة الحرارية الكلية :

$$\int \left(7-\beta+\frac{\ell^2}{2}\right)dt = \left(7-\beta+\frac{\ell^2}{2}\right)t + C$$

$$u(x,0) = x+1 \Rightarrow C = \int_0^{\ell} (x+1)\,dx$$

$$= \left[\frac{x^2}{2}+x\right]_0^{\ell} = \frac{\ell^2}{2}+\ell$$

$$E(t) = \left(7+\beta+\frac{\ell^2}{2}\right)t + \frac{\ell^2}{2} + \ell$$

السؤال الثالث:

حل بطريقة فصل المتغيرات المسائل التالية:

1

$$U_t = U_{xx}$$

$$U(x,0) = 3\sin\left(\frac{\pi}{2}x\right) - \sin\left(\frac{3\pi}{2}x\right)$$

$$U(0,t) = U(2,t) = 0$$

الحل: نفرض أن لها حلاً من الشكل:

$$U(x,t) = X(x)T(t)$$

$$U_t(x,t) = X(x)T'(t)$$

$$U_{xx}(x,t) = X''(x)T(t)$$

$$X(x)T'(t) = X''(x)T(t)$$

$$X(x)T'(t) = X''(x)T(t)$$

$$\underbrace{\frac{T'(t)}{T(t)}}_{\text{تابع لـ } t \text{ فقط}} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{تابع لـ } x \text{ فقط}} = \lambda$$

يصبح لدينا المعادلتين:

$$T'(t) - \lambda T(t) = 0$$

$$X''(x) - \lambda X(x) = 0$$

$$U(0,t) = X(0)\underbrace{T(t)}_{\neq 0} = 0 \Rightarrow X(0) = 0$$

$$U(2,t) = X(2)\underbrace{T(t)}_{\neq 0} = 0 \Rightarrow X(2) = 0$$

نبدأ بحل المعادلة:

$$X''(x) - \lambda X(x) = 0$$

$$m^2 - \lambda = 0$$

نناقش حسب قيم λ:

① $\lambda = 0 \Leftarrow$

$$X(x) = Ax + B$$

$$\left.\begin{array}{l} X(0) = \boxed{B = 0} \\ X(2) = 2A = 0 \Rightarrow \boxed{A = 0} \end{array}\right\}$$

حل مرفوض

② $\lambda > 0 \Leftarrow$

$$X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$$

$$X(0) = A + B = 0 \Rightarrow \boxed{A = -B}$$

$$X(2) = A(e^{2\sqrt{\lambda}} - e^{-2\sqrt{\lambda}}) = 0 \Rightarrow \left.\begin{array}{l} A = 0 \\ B = 0 \end{array}\right\}$$

حل مرفوض

③ $\lambda < 0 \Leftarrow$

$$X(x) = A\cos\sqrt{-\lambda}\,x + B\sin\sqrt{-\lambda}\,x$$

$$X(0) = A = 0$$

$$X(2) = \underbrace{B}_{\neq 0}\sin 2\sqrt{-\lambda} = 0 \Rightarrow \sin 2\sqrt{-\lambda} = 0$$

$$2\sqrt{-\lambda} = n\pi \Rightarrow \sqrt{-\lambda} = \frac{n\pi}{2}$$

$$\Rightarrow \lambda = -\frac{n^2\pi^2}{4}$$

$$X_n(x) = B_n \sin\frac{n\pi}{2}x$$

$$T_n(t) = ce^{-\frac{n^2\pi^2}{4}t}$$

$$U_n(x,t) = D_n e^{-\frac{n^2\pi^2}{4}t}\sin\frac{n\pi}{2}x$$

$$u(x,t) = \sum_{n=1}^{\infty} D_n e^{-\frac{n^2\pi^2}{4}t} \sin\frac{n\pi}{2}x$$

$$u(x,0) = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi}{2}x = 3\sin\left(\frac{\pi}{2}x\right) - \sin\left(\frac{3\pi}{2}x\right)$$

$$= \frac{2}{2}\int_0^2 \left(3\sin\left(\frac{\pi}{2}x\right) - \sin\left(\frac{3\pi}{2}x\right)\right)\sin\frac{n\pi}{2}x \, dx$$

$$= \int_0^2 \left(3\sin\left(\frac{\pi}{2}x\right)\sin\left(\frac{n\pi}{2}x\right) - \sin\left(\frac{3\pi}{2}x\right)\sin\left(\frac{n\pi}{2}x\right)\right)dx$$

$$= \frac{1}{2}\int_0^2 \Big[3\left(\cos\left(\frac{1-n}{2}\pi\right)x - \cos\left(\frac{(1+n)\pi}{2}x\right)\right) - \cos\left(\frac{3-n}{2}\pi\right)x +$$

$$\cos\left(\frac{3+n}{2}\pi\right)x\Big]\, dx$$

$$= \frac{1}{2}\Big[\frac{3\times 2}{(1-n)\pi}\sin\left(\frac{1-n}{2}\pi\right)x - \frac{3\times 2}{(1+2)\pi}\sin\left(\frac{1+n}{2}\pi\right)x -$$

$$\frac{2}{(3-n)\pi}\sin\left(\frac{3-n}{2}\pi\right)x + \frac{2}{(3+n)\pi}\sin\left(\frac{3+n}{2}\pi\right)x\Big]_0^2$$

$D_n = 0$ ، $n \neq \{1, 3\}$

لما $n=1 \Leftarrow$

$$D_1 = \int_0^2 \left(3\sin\left(\frac{\pi}{2}x\right)\sin\left(\frac{\pi}{2}x\right) - \sin\left(\frac{3\pi}{2}x\right)\sin\left(\frac{\pi}{2}x\right)\right)dx$$

$$= \int_0^2 \left(3\sin^2\left(\frac{\pi}{2}\right)x - \frac{1}{2}\cos(\pi x) + \frac{1}{2}\cos 2\pi x\right)dx$$

$$\int_0^2 \left(\frac{3}{2}(1-\cos \pi x) - \frac{1}{2}\cos \pi x + \frac{1}{2}\cos 2\pi x \right) dx$$

$$D_1 = \frac{1}{2}\left[3x - \frac{3}{\pi}\sin \pi x - \frac{1}{\pi}\sin \pi x + \frac{1}{2\pi}\sin 2\pi x\right]_0^2$$

$$D_1 = 3 - 0 = 3 \Rightarrow \boxed{D_1 = 3}$$

عندما $n=3$

$$D_3 = \int_0^2 \left(3\sin\left(\frac{\pi}{2}x\right)\sin\left(\frac{3\pi}{2}x\right) - \sin^2\left(\frac{3\pi}{2}\right)x\right) dx$$

$$D_3 = \frac{1}{2}\int_0^2 (3\cos \pi x - \cos 2\pi x - 1 + \cos 3\pi x)\, dx$$

$$D_3 = \frac{1}{2}\left[\frac{3}{\pi}\sin \pi x - \frac{1}{2\pi}\sin 2\pi x - x + \frac{1}{3\pi}\sin 3\pi x\right]_0^2$$

$$D_3 = -1 - 0 = -1 \Rightarrow \boxed{D_3 = -1}$$

$$U(x,t) = 3e^{-\frac{\pi^2}{4}t}\sin\frac{\pi}{2}x + \left(-e^{-\frac{9\pi^2}{4}t}\sin\frac{3\pi}{2}x\right)$$

$$U(x,t) = 3e^{-\frac{\pi^2}{4}t}\sin\frac{\pi}{2}x - e^{-9\frac{\pi^2}{4}t}\sin\frac{3\pi}{2}x$$

طرح لإيجاد D_n لدينا:

$$U(x,0) = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi}{2}x = 3\sin\frac{\pi}{2}x - \sin\frac{3\pi}{2}x$$

بالمطابقة بين الطرفين نجد أنّ:

$D_1 = 3$

$D_3 = -1$

$D_n = 0 \; ; \; n \neq \{1, 3\}$

وبالتالي:

$$u(x,t) = 3e^{-\frac{\pi^2}{4}t}\sin\frac{\pi}{2}x - e^{-\frac{9\pi^2}{4}t}\sin\frac{3\pi}{2}x$$

الهذه الطريقة لا تطبق إلا إذا كان الطرف الثاني من أمثال $\sin x$ أو $\cos x$ حتى نتمكن من إجراء المطابقة.

(2)

$$u_t = \frac{1}{4}u_{xx}$$

$$u(x,0) = 100x(1-x)$$

$$u_x(0,t) = u_x(\ell,t) = 0$$

$$\ell = 1$$

الحل: نفرض أن لها حلاً من الشكل:

$$u(x,t) = X(x)T(t)$$

$$u_t(x,t) = X(x)T'(t)$$

$$u_{xx}(x,t) = X''(x)T(t)$$

$$X(x)T'(t) = X''(x)T(t) \Rightarrow \underbrace{\frac{T'(t)}{T(t)}}_{\text{تابع لـ } t \text{ فقط}} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{تابع لـ } x \text{ فقط}} = \lambda$$

بالتالي نحصل على المعادلتين:

$$T'(t) - \lambda T(t) = 0$$

$$X''(x) - \lambda X(x) = 0$$

$$u_x(0,t) = X'(0)\underbrace{T(t)}_{\neq 0} = 0 \Rightarrow X'(0) = 0$$

$U_x(1,t) = X'(1)\underbrace{T(t)}_{\neq 0} = 0 \Rightarrow X'(1) = 0$

نبدأ بحل المسألة :

$X''(x) - \lambda X(x) = 0$

$m^2 - \lambda = 0$

نناقش حسب قيم λ :

1) $\lambda = 0 \Leftarrow$

$X(x) = Ax + B$

$X'(x) = A \Rightarrow X'(0) = A = 0$

$X'(1) = A = 0$

$\Rightarrow X(x) = B$

2) $\lambda > 0 \Leftarrow$

$X(x) = A_1 e^{\sqrt{\lambda}x} + B_1 e^{-\sqrt{\lambda}x}$

$X'(x) = \sqrt{\lambda} A_1 e^{\sqrt{\lambda}x} - \sqrt{\lambda} B_1 e^{-\sqrt{\lambda}x}$

$X'(0) = \underbrace{\sqrt{\lambda}}_{\neq 0}(A_1 - B_1) = 0 \Rightarrow \boxed{A_1 = B_1}$

$X'(1) = \sqrt{\lambda} A_1 (e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}}) = 0 \Rightarrow \left.\begin{aligned} A_1 &= 0 \\ B_1 &= 0 \end{aligned}\right\}$

حل مرفوض

3) $\lambda < 0 \Leftarrow$

$X(x) = A\cos\sqrt{-\lambda}\,x + B\sin\sqrt{-\lambda}\,x$

$X'(x) = -\sqrt{-\lambda}\,A\sin\sqrt{-\lambda}\,x + B\sqrt{-\lambda}\cos\sqrt{-\lambda}\,x$

$X'(0) = B\underbrace{\sqrt{-\lambda}}_{0\neq} = 0 \Rightarrow \boxed{B = 0}$

$X'(1) = -\sqrt{-\lambda}\, A \sin\sqrt{-\lambda} = 0 \Rightarrow \sin\sqrt{-\lambda} = 0 \Rightarrow \sqrt{-\lambda} = n\pi \Rightarrow$

$\lambda = -n^2\pi^2$

$X_n(x) = A_n \cos n\pi x$

$T(t) = C e^{-n^2\pi^2 t}$

$U_n(x,t) = D_n e^{-n^2\pi^2 t} \cos n\pi x$

$$u(x,t) = \sum_{n=0}^{\infty} D_n e^{-n^2\pi^2 t} \cos n\pi x$$

$$u(x,0) = \sum_{n=0}^{\infty} D_n \cos n\pi x = 100x(1-x)$$

$$D_0 = \int_0^1 100x(1-x)\,dx = \int_0^1 (100x - 100x^2)\,dx$$

$$= \left[50x^2 - \frac{100}{3}x^3\right]_0^1 = 50 - \frac{100}{3} = \frac{50}{3} \Rightarrow \boxed{D_0 = \frac{50}{3}}$$

$$\Rightarrow D_n = \int_0^1 (100x(1-x)\cos n\pi x)\,dx$$

تكامل بالتجزئة :

$u = 100x(1-x)$

$\Rightarrow du = 100 - 200x = 100(1-2x)$

$dv = \cos n\pi x\,dx$

$\Rightarrow v = \frac{1}{n\pi} \sin n\pi x$

$$D_n = \left[100x(1-x)\frac{1}{n\pi}\sin n\pi x\right]_0^1 - \int_0^1 100(1-2x)\frac{1}{n\pi}\sin n\pi x\,dx$$

$$D_n = \frac{-100}{n\pi}\int_0^1 (1-2x)\sin n\pi x \, dx$$

$u = 1-2x \qquad dv = \sin n\pi x \, dx$

$du = -2 \qquad v = \frac{-1}{n\pi}\cos n\pi x$

$$D_n = \frac{100}{n^2\pi^2}\left[\left[(1-2x)\cos n\pi x\right]_0^1 + 2\int_0^1 \cos n\pi x \, dx\right]$$

$$= \frac{100}{n^2\pi^2}\left[\left[-\cos \pi n - 1\right] + \frac{2}{n\pi}\left[\underbrace{\sin n\pi x}_{=0}\right]_0^1\right]$$

$$= \frac{100}{n^2\pi^2}\left(-(-1)^n - 1\right) = \frac{100}{n^2\pi^2}\left((-1)^{n+1} - 1\right) = \frac{100}{n^2\pi^2}\left((-1)^{n+1} - 1\right)$$

$$u(x,t) = \sum_{n=0}^{\infty} D_n e^{-n^2\pi^2 t}\cos n\pi x$$

$$u(x,t) = \frac{50}{3} + \sum_{n=1}^{\infty}\frac{100}{n^2\pi^2}\left((-1)^{n+1} - 1\right)e^{-n^2\pi^2 t}\cos n\pi x$$

وهو الحل المطلوب

انتهت المحاضرة