كلية العلوم

القسم : **الرياضيات**

السنة : **الثانية**

المادة : معادلات تفاضلية١

المحاضرة : السادسة/عملي/

{{ A to Z مكتبة }}

Facebook Group : A to Z مكتبة

التمرين: حل المعادلات التفاضلية الآتية

① $y'(2y-y') = y^2\sin^2 x$

$2yy' - y'^2 = y^2\sin x$

محلولة بالنسبة لـ P ⟸ نفرض ⟸ $y' = P$

$2yP - P^2 = y^2\sin x$

$\cancel{P^2 = 2yP}$ $2yP - P^2 - y^2\sin x = 0$

$P^2 - 2yP + y^2\sin x = 0$

$\Delta = b^2 - 4ac \Rightarrow \Delta = 4y^2P^2 - 4y^2P^2\sin^2 x = 0$

$= 4y^2P^2(1-\sin^2 x) = 4y^2P^2(\cos^2 x)$

$$P_1 = \frac{2y + 2y\cos x}{2} = y(1+\cos x)$$

$y' = y(1+\cos x)$

$\frac{dy}{y} = (1+\cos x)dx \Rightarrow \ln y = x + \sin x + c_1$

$\ln y - x - \sin x = c_1$

$$P_2 = \frac{2y - 2y\cos x}{2} = y(1-\cos x)$$

$y' = y(1-\cos x) \Rightarrow \frac{dy}{y} = (1-\cos x)\,dx$

$\ln y = x - \sin x + c_1$

$\ln y - x + \sin x - c_1 = 0$

$(\ln y - x - \sin x - c_1)(\ln y - x + \sin x - c_1) = 0$

② $y = 3Px + 6y^2P^2$

$3Px = y - 6y^2P^2 \Rightarrow \boxed{x = \frac{y}{3P} - 2y^2P}$ --- ①

نفاضله بالنسبة لـ y

$\frac{dx}{dy} = \frac{1}{P} = \frac{1}{3P} - 4yP\left(-\frac{y}{3P^2} - 2y^2\right)\frac{dP}{dy}$

$-\frac{1}{3P} + \frac{1}{P} + 4yP = \left(-\frac{y}{3P^2} - 2y^2\right)\frac{dP}{dy}$

$\frac{2}{3P} + 4yP = -\frac{y}{P}\left(\frac{1}{3P} + 2yP\right)\frac{dP}{dy}$

$2 = -\frac{y}{P}\frac{dP}{dy}$

$-\frac{dP}{P} = \frac{dy}{y} \Rightarrow -\frac{1}{2}\ln P = \ln y + \ln c$

$-\frac{1}{\sqrt{P}} = cy \Rightarrow y = -\frac{1}{c\sqrt{P}}$ --- ②

① و ② يشكلان الحل الوسيطي

③. $xP^2-2yP+4x=0$

$$2yP=xP^2+4x$$

$$y=\frac{xP^2+4x}{2P}$$

$$\boxed{y=\frac{1}{2}xP+2\frac{x}{P}} \quad ---- (1)$$

نفاضله بالنسبة لـ x

$$\frac{dy}{dx}=P=\frac{1}{2}P+\frac{2}{P}+\left(\frac{1}{2}x-\frac{2x}{P^2}\right)\times\frac{dP}{dx}$$

$$\frac{1}{2}P-\frac{2}{P}=x\left(\frac{1}{2}-\frac{2}{P^2}\right)\frac{dP}{dx}$$

$$\frac{P^2-2}{2P}=x\left(\frac{P^2-2}{2P^2}\right)\frac{dP}{dx}$$

$$1=\frac{x}{P}\frac{dP}{dx} \Rightarrow \frac{dx}{x}=\frac{dP}{P}$$

$$\ln x+\ln c=\ln P$$

$$\boxed{cx=P}$$

لايجاد الحل العام نعوضه (2) في (1)

$$y=\frac{c}{2}x^2+\frac{2}{c}$$

④. $(y')^3+2x(y')^2=y$

نفرضه $y'=P$

$$P^3+2xP=y$$

$$\frac{dy}{dx} = P = 2P^2 + (3P^2 + 4xP)\frac{dP}{dx}$$

$$P - 2P^2 = (3P^2 + 4xP)\frac{dP}{dx}$$

$$(1-2P) = (3P + 4x)\frac{dP}{dx}$$

$$(1-2P)\frac{dx}{dP} = 3P + 4x$$

$$x' - \frac{4}{1-2P}x = \frac{3P}{1-2P}$$

$$\mu = e^{\int \frac{-4}{1-2P}dP} = e^{2\ln(1-2P)} = (1-2P)^2$$

$$\underbrace{x'(1-2P)^2 - 4(1-2P)x}_{\text{زمرة}} = 3P(1-2P)$$

$$(1-2P)^2 x = \frac{3}{2}P^2 - 2P^3 + C$$

$$x = \frac{3P^2}{2(1-2P)^2} - 2\frac{P^3}{(1-2P)^2} + \frac{C}{(1-2P)^2}$$

$\boxed{y = P^3 + 2xP^2}$ الحلة الوسيطي ⟸

(5) $y = xy' + y'^2$

حل:

معادلة كليرو حلها:

$y = xc + c^2$

(6) $x^2y'' + 2xy' - 1 = 0$

$y' = Z \Rightarrow y'' = Z'$

$x^2 Z' + 2xZ - 1 = 0$

$x^2 dZ + 2xZ dx - dx = 0$

$x^2 Z + x = c$

$Z = \frac{c - x}{x^2} = \frac{c}{x^2} - \frac{1}{x}$

$y' = \frac{c}{x^2} - \frac{1}{x} \Rightarrow y = -\frac{c}{x} - \ln x + c_1$

(7) $yy'' + (y')^2$

نفرض $y' = Z \Rightarrow y'' = Z.Z'$

$yZ.Z' + Z^2 = 0$

$yZ \frac{dZ}{dy} + Z^2 = 0$

$yZ dZ + Z^2 dy = 0$

نقسم على $Zy^2 \Leftarrow$

$\frac{1}{Z} dZ + \frac{dy}{y} = 0$

$\ln(Z) + \ln(y) = \ln(c)$

$Z.y = c$

ولدينا $y' = Z \Leftarrow$

$$y'y = c$$

$$y\,dy = c\,dx \Rightarrow \boxed{y^2 = 2cx}$$

(8). $y''' + x(y'')^2 = 0$

$$y'' = Z \Rightarrow y''' = Z'$$

$$Z' + xZ^2 = 0$$

$$\frac{dZ}{dx} + xZ^2 = 0 \Rightarrow dZ + xZ^2 dx$$

نقسم على Z^2

$$\frac{dZ}{Z^2} + x\,dx = 0 \Rightarrow -\frac{1}{Z} + \frac{x^2}{2} + \frac{c}{2} = 0$$

$$-\frac{1}{Z} + \frac{x^2}{2} + \frac{c}{2} = 0$$

$$Z = \frac{2}{x^2 + c}$$

$$y'' = \frac{2}{x^2+c} \Rightarrow y' = \frac{2}{\sqrt{c}} \arctan\left(\frac{x}{\sqrt{c}}\right) + c_1$$

بالمكاملة بالتجزئة نجد أنَّ

$$y = \frac{2x}{\sqrt{c}} \arctan\left(\frac{x}{\sqrt{c}}\right) \ln(x^2+c) + c_1 x + c_2$$

(9). $yy'' + (y')^2 = 0$

$$Z = \frac{y'}{y}$$

$$\frac{y''}{y} + \left(\frac{y'}{y}\right)^2 = 0$$

$$Z^2 + Z' + Z^2 = 0$$

$$2Z^2 + Z' = 0$$

$$2Z^2 + \frac{dZ}{dx} = 0 \Longrightarrow + \frac{dZ}{2Z^2} + dx = -\frac{1}{2Z} + x = -c$$

$$\frac{1}{2Z} = x + c \Longrightarrow 2Z = \frac{1}{x+c} \Longrightarrow$$

$$2\frac{y'}{y} = \frac{1}{x+c} \Longrightarrow 2\frac{dy}{y} = \frac{dx}{x+c}$$

$$2\ln(y) = \ln(x+c) + \ln c_1$$

$$y^2 = (x+c)c_1$$

10. $xy'' + y' - \frac{6}{x}y = 0$.

متجانسة بالنسبة لـ x

نفرض $x = e^t$

$$xy' = y'_t$$

$$x^2y'' = y''_t - y'_t$$

$$x^2y'' + xy' - 6y = 0$$

$$y''_t - y'_t + y'_t - 6y = 0$$

$$y''_t - 6y = 0$$

فالية منه المتحول $\longleftarrow T$

$$y'' = Z'Z_y \Longleftarrow y' = Z$$

$Z'Z - 6y = 0$

$ZdZ - 6ydy = 0$

$\frac{Z^2}{2} - 3y^2 = C_1 \Rightarrow \frac{Z^2}{2} = 2y^3 + C_1$

$Z^2 = 6y^2 + 4C_1 \Rightarrow Z = \sqrt{6y^2 + 4C_1}$

$y' = \sqrt{6y^2 + a^2} \;;\; a^2 = 4C_1$

$\frac{dy}{\sqrt{6y^2 + a^2}} = dt$

$\frac{1}{\sqrt{6}a} \arg sh(\sqrt{6}y) = t + C_2$

$\frac{1}{\sqrt{6}a} \arg sh(\sqrt{6}y) = \ln x + C_2$

(11). $xy'' = y' - \frac{y}{x}$

نفرض منه $Z = \frac{y}{x}$

$y' = xZ' + Z$

$y'' = 2Z' + xZ''$

$x(2Z' + xZ'') = xZ' + \cancel{Z} - \cancel{Z}$

$xZ' + x^2Z'' = 0$

$\Leftarrow x = e^t$

$xZ' = Z'_t$

$x^2Z'' = Z''_t - Z'_t$

$Z'_t - Z''_t + Z'_t = 0$

$Z''_t = 0 \Rightarrow Z'_t = C_1 \Rightarrow Z = C_1t + C_2$

$$\frac{y}{x} = c_1 \ln x + c_2$$

12. $x = y'''^2 + 1$

نفرض $y''' = T$

$x = T^2 + 1$

$dx = 2T\,dT$

$dy'' = y'''\,dx$

$= T(2T\,dT) \Rightarrow dy'' = \frac{2}{3}T^3 + c_1$

$dy' = y''\,dx = (\frac{4}{3}T^4 + 2c_1T)\,dT$

$y' = \frac{4T^5}{15} + c_1T^2 + c_2$

$dy = y'\,dx$

$dy = (\frac{8T^6}{15} + 2c_1T^3 + 2c_2T)\,dT$

$y = \frac{8T^7}{105} + c_1\frac{T^4}{2} + c_2T^2 + c_3$

الحل الوسيطي

$x = T^2 + 1$

$y = \frac{8T^7}{105} + c_1\frac{T^4}{2} + c_2T^2 + c_3$

انتهت المحاضرة