كلية العلوم

القسم : **الرياضيات**

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1 Numbers

§1. THE INTEGERS

The most common numbers are those used for counting, namely the numbers

$$1, 2, 3, 4, \ldots,$$

which are called the **positive integers.** Even for counting, we need at least one other number, namely,

0 (**zero**).

For instance, we may wish to count the number of right answers you may get on a test for this course, out of a possible 100. If you get 100, then all your answers were correct. If you get 0, then no answer was correct.

The positive integers and zero can be represented geometrically on a line, in a manner similar to a ruler or a measuring stick:

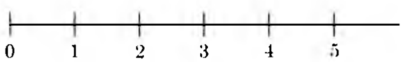


Fig. 1–1

For this we first have to select a unit of distance, say the inch, and then on the line we mark off the inches to the right as in the picture.

For convenience, it is useful to have a name for the positive integers together with zero, and we shall call these the **natural numbers.** Thus 0 is a natural number, so is 2, and so is 124,521. The natural numbers can be used to measure distances, as with the ruler.

By definition, the point represented by 0 is called the **origin.**

The natural numbers can also be used to measure other things. For example, a thermometer is like a ruler which measures temperature. However, the thermometer shows us that we encounter other types of numbers besides the natural numbers, because there may be temperatures which may go below 0. Thus we encounter naturally what we shall call **negative integers** which we call **minus** 1, **minus** 2, **minus** 3, . . . , and which we write as

$$-1, \ -2, \ -3, \ -4, \ldots .$$

We represent the negative integers on a line as being on the other side of 0 from the positive integers, like this:

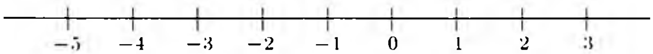


Fig. 1–2

The positive integers, negative integers, and zero all together are called the **integers.** Thus -9, 0, 10, -5 are all integers.

If we view the line as a thermometer, on which a unit of temperature has been selected, say the degree Fahrenheit, then each integer represents a certain temperature. The negative integers represent temperatures below zero.

Our discussion is already typical of many discussions which will occur in this course, concerning mathematical objects and their applicability to physical situations. In the present instance, we have the integers as mathematical objects, which are essentially abstract quantities. We also have different applications for them, for instance measuring distance or temperatures. These are of course not the only applications. Namely, we can use the integers to measure time. We take the origin 0 to represent the year of the birth of Christ. Then the positive integers represent years after the birth of Christ (called AD years), while the negative integers can be used to represent BC years. With this convention, we can say that the year -500 is the year 500 BC.

Adding a positive number, say 7, to another number, means that we must move 7 units to the right of the other number. For instance,

$$5 + 7 = 12.$$

Seven units to the right of 5 yields 12. On the thermometer, we would of course be moving upward instead of right. For instance, if the temperature at a given time is 5° and if it goes up by 7°, then the new temperature is 12°.

Observe the very simple rule for addition with 0, namely

N1. $$\boxed{0 + a = a + 0 = a}$$

for any integer a.

What about adding negative numbers? Look at the thermometer again. Suppose the temperature at a given time is 10°, and the temperature drops by 15°. The new temperature is then $-5°$, and we can write

$$10 - 15 = -5.$$

Thus -5 is the result of subtracting 15 from 10, or of adding -15 to 10.

In terms of points on a line, adding a negative number, say -3, to another number means that we must move 3 units to the left of this other number. For example,

$$5 + (-3) = 2$$

because starting with 5 and moving 3 units to the left yields 2. Similarly,

$$7 + (-3) = 4, \quad \text{and} \quad 3 + (-5) = -2.$$

Note that we have

$$3 + (-3) = 0 \quad \text{or} \quad 5 + (-5) = 0.$$

We can also write these equations in the form

$$(-3) + 3 = 0 \quad \text{or} \quad (-5) + 5 = 0.$$

For instance, if we start 3 units to the left of 0 and move 3 units to the right, we get 0. Thus, in general, we have the formulas (by assumption):

N2. $$a + (-a) = 0 \quad \text{and also} \quad -a + a = 0.$$

In the representation of integers on the line, this means that a and $-a$ lie on opposite sides of 0 on that line, as shown on the next picture:

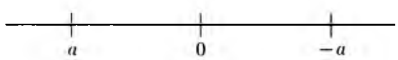
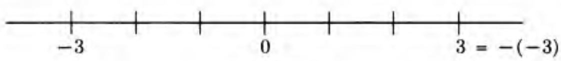


Fig. 1–3

Thus according to this representation we can now write

$$3 = -(-3) \quad \text{or} \quad 5 = -(-5).$$

In these special cases, the pictures are:



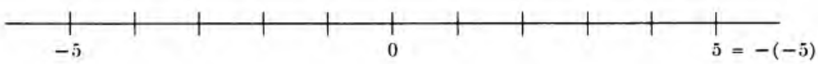


Fig. 1–4

Remark. We use the name

$$\textbf{minus } a \qquad \text{for} \qquad -a$$

rather than the words "negative a" which have found some currency recently. I find the words "negative a" confusing, because they suggest that $-a$ is a negative number. This is not true unless a itself is positive. For instance,

$$3 = -(-3)$$

is a positive number, but 3 is equal to $-a$, where $a = -3$, and a is a negative number.

Because of the property

$$a + (-a) = 0,$$

one also calls $-a$ the **additive inverse** of a.

The sum and product of integers are also integers, and the next sections are devoted to a description of the rules governing addition and multiplication.

§2. RULES FOR ADDITION

Integers follow very simple rules for addition. These are:

Commutativity. *If a, b are integers, then*

$$\boxed{a + b = b + a.}$$

For instance, we have

$$3 + 5 = 5 + 3 = 8,$$

or in an example with negative numbers, we have

$$-2 + 5 = 3 = 5 + (-2).$$

Associativity. *If a, b, c are integers, then*

$$\boxed{(a + b) + c = a + (b + c).}$$

In view of this, it is unnecessary to use parentheses in such a simple context, and we write simply

$$a + b + c.$$

For instance,

$$(3 + 5) + 9 = 8 + 9 = 17,$$
$$3 + (5 + 9) = 3 + 14 = 17.$$

We write simply

$$3 + 5 + 9 = 17.$$

Associativity also holds with negative numbers. For example,

$$(-2 + 5) + 4 = 3 + 4 = 7,$$
$$-2 + (5 + 4) = -2 + 9 = 7.$$

Also,

$$(2 + (-5)) + (-3) = -3 + (-3) = -6,$$
$$2 + (-5 + (-3)) = 2 + (-8) = -6.$$

The rules of addition mentioned above will not be proved, but we shall prove other rules from them.

To begin with, note that:

N3. *If $a + b = 0$, then $b = -a$ and $a = -b$.*

To prove this, add $-a$ to both sides of the equation $a + b = 0$. We get

$$-a + a + b = -a + 0 = -a.$$

Since $-a + a + b = 0 + b = b$, we find

$$b = -a$$

as desired. Similarly, we find $a = -b$. We could also conclude that

$$-b = -(-a) = a.$$

As a matter of convention, we shall write

$$a - b$$

instead of

$$a + (-b).$$

Thus a sum involving three terms may be written in many ways, as follows:

$$\begin{aligned} (a - b) + c &= (a + (-b)) + c \\ &= a + (-b + c) && \text{by associativity} \\ &= a + (c - b) && \text{by commutativity} \\ &= (a + c) - b && \text{by associativity,} \end{aligned}$$

and we can also write this sum as

$$a - b + c = a + c - b,$$

omitting the parentheses. Generally, in taking the sum of integers, we can take the sum in any order by applying associativity and commutativity repeatedly.

As a special case of **N3,** for any integer a we have

N4.

$$\boxed{a = -(-a).}$$

This is true because

$$a + (-a) = 0,$$

and we can apply **N3** with $b = -a$. Remark that this formula is true whether a is positive, negative, or 0. If a is positive, then $-a$ is negative. If a is negative, then $-a$ is positive. In the geometric representation of numbers on the line, a and $-a$ occur symmetrically on the line on opposite sides of 0. Of course, we can pile up minus signs and get other relationships, like

$$-3 = -(-(-3)),$$

or

$$3 = -(-3) = -(-(-(-3))).$$

Thus when we pile up the minus signs in front of a, we obtain a or $-a$ alternatively. For the general formula with the appropriate notation, cf. Exercises 5 and 6 of §4.

From our rules of operation we can now prove:

For any integers a, b we have

$$-(a + b) = -a + (-b)$$

or, in other words,

N5.
$$-(a + b) = -a - b.$$

Proof. Remember that if x, y are integers, then $x = -y$ and $y = -x$ mean that $x + y = 0$. Thus to prove our assertion, we must show that

$$(a + b) + (-a - b) = 0.$$

But this comes out immediately, namely,

$$\begin{aligned}(a + b) + (-a - b) &= a + b - a - b && \text{by associativity}\\ &= a - a + b - b && \text{by commutativity}\\ &= 0 + 0\\ &= 0.\end{aligned}$$

This proves our formula.

Example. We have

$$\begin{aligned}-(3 + 5) &= -3 - 5 = -8,\\ -(-4 + 5) &= -(-4) - 5 = 4 - 5 = -1,\\ -(3 - 7) &= -3 - (-7) = -3 + 7 = 4.\end{aligned}$$

You should be very careful when you take the negative of a sum which involves itself in negative numbers, taking into account that

$$-(-a) = a.$$

The following rule concerning positive integers is so natural that you probably would not even think it worth while to take special notice of it. We still state it explicitly.

If a, b are positive integers, then $a + b$ is also a positive integer.

For instance, 17 and 45 are positive integers, and their sum, 62, is also a positive integer.

We assume this rule concerning positivity. We shall see later that it also applies to positive real numbers. From it we can prove:

If a, b are negative integers, then $a + b$ is negative.

Proof. We can write $a = -n$ and $b = -m$, where m, n are positive. Therefore

$$a + b = -n - m = -(n + m),$$

which shows that $a + b$ is negative, because $n + m$ is positive.

Example. If we have the relationship between three numbers

$$\boxed{a + b = c,}$$

then we can derive other relationships between them. For instance, add $-b$ to both sides of this equation. We get

$$a + b - b = c - b,$$

whence $a + 0 = c - b$, or in other words,

$$\boxed{a = c - b.}$$

Similarly, we conclude that

$$\boxed{b = c - a.}$$

For instance, if

$$x + 3 = 5,$$

then

$$x = 5 - 3 = 2.$$

If

$$4 - a = 3,$$

then adding a to both sides yields

$$4 = 3 + a,$$

and subtracting 3 from both sides yields

$$1 = a.$$

If

$$-2 - y = 5,$$

then

$$-7 = y \qquad \text{or} \qquad y = -7.$$

EXERCISES

Justify each step, using commutativity and associativity in proving the following identities.

1. $(a + b) + (c + d) = (a + d) + (b + c)$
2. $(a + b) + (c + d) = (a + c) + (b + d)$
3. $(a - b) + (c - d) = (a + c) + (-b - d)$
4. $(a - b) + (c - d) = (a + c) - (b + d)$
5. $(a - b) + (c - d) = (a - d) + (c - b)$
6. $(a - b) + (c - d) = -(b + d) + (a + c)$
7. $(a - b) + (c - d) = -(b + d) - (-a - c)$
8. $((x + y) + z) + w = (x + z) + (y + w)$
9. $(x - y) - (z - w) = (x + w) - y - z$
10. $(x - y) - (z - w) = (x - z) + (w - y)$
11. Show that $-(a + b + c) = -a + (-b) + (-c)$.
12. Show that $-(a - b - c) = -a + b + c$.
13. Show that $-(a - b) = b - a$.

Solve for x in the following equations.

14. $-2 + x = 4$
15. $2 - x = 5$
16. $x - 3 = 7$
17. $-x + 4 = -1$
18. $4 - x = 8$
19. $-5 - x = -2$
20. $-7 + x = -10$
21. $-3 + x = 4$

22. Prove the **cancellation law for addition:**

$$\boxed{\text{If } a + b = a + c, \text{ then } b = c.}$$

23. Prove: If $a + b = a$, then $b = 0$.

§3. RULES FOR MULTIPLICATION

We can multiply integers, and the product of two integers is again an integer. We shall list the rules which apply to multiplication and to its relations with addition.

We again have the rules of ***commutativity*** and ***associativity:***

$$\boxed{ab = ba} \quad \text{and} \quad \boxed{(ab)c = a(bc).}$$

We emphasize that these apply whether a, b, c are negative, positive, or zero. Multiplication is also denoted by a dot. For instance

$$3 \cdot 7 = 21,$$

and

$$(3 \cdot 7) \cdot 4 = 21 \cdot 4 = 84,$$
$$3 \cdot (7 \cdot 4) = 3 \cdot 28 = 84.$$

For any integer a, the rules of multiplication by 1 *and* 0 *are:*

N6. $$\boxed{1a = a} \quad \textit{and} \quad \boxed{0a = 0.}$$

Example. We have

$$\begin{aligned}(2a)(3b) &= 2(a(3b)) \\ &= 2(3a)b \\ &= (2 \cdot 3)ab \\ &= 6ab.\end{aligned}$$

In this example we have done something which is frequently useful, namely we have moved to one side all the explicit numbers like 2, 3 and put on the other side those numbers denoted by a letter like a or b. Using commutativity and associativity, we can prove similarly

$$(5x)(7y) = 35xy$$

or, with more factors,

$$(2a)(3b)(5x) = 30abx.$$

We suggest that you carry out the proof of this equality completely, using associativity and commutativity for multiplication.

Finally, we have the rule of ***distributivity***, namely

$$\boxed{a(b + c) = ab + ac}$$

and also on the other side,

$$\boxed{(b + c)a = ba + ca.}$$

These rules will not be proved, but will be used constantly. We shall, however, make some comments on them, and prove other rules from them.

First observe that if we just assume distributivity on one side, and commutativity, then we can prove distributivity on the other side. Namely, assuming distributivity on the left, we have

$$(b + c)a = a(b + c) = ab + ac = ba + ca,$$

which is the proof of distributivity on the right.

Observe also that our rule $0a = 0$ can be proved from the other rules concerning multiplication and the properties of addition. We carry out the proof as an example. We have

$$0a + a = 0a + 1a = (0 + 1)a = 1a = a.$$

Thus

$$0a + a = a.$$

Adding $-a$ to both sides, we obtain

$$0a + a - a = a - a = 0.$$

The left-hand side is simply

$$0a + a - a = 0a + 0 = 0a,$$

so that we obtain $0a = 0$, as desired.

We can also prove

N7. $$\boxed{(-1)a = -a.}$$

Proof. We have

$$(-1)a + a = (-1)a + 1a = (-1 + 1)a = 0a = 0.$$

By definition, $(-1)a + a = 0$ means that $(-1)a = -a$, as was to be shown.

We have

N8. $$\boxed{-(ab) = (-a)b.}$$

Proof. We must show that $(-a)b$ is the negative of ab. This amounts to showing that

$$ab + (-a)b = 0.$$

But we have by distributivity

$$ab + (-a)b = (a + (-a))b = 0b = 0,$$

thus proving what we wanted.

Similarly, we leave to the reader the proof that

N9. $$\boxed{-(ab) = a(-b).}$$

Example. We have

$$-(3a) = (-3)a = 3(-a).$$

Also,

$$4(a - 5b) = 4a - 20b.$$

Also,

$$-3(5a - 7b) = -15a + 21b.$$

In each of the above cases, you should indicate specifically each one of the rules we have used to derive the desired equality. Again, we emphasize that you should be especially careful when working with negative numbers and repeated minus signs. This is one of the most frequent sources of error when we work with multiplication and addition.

Example. We have

$$\begin{aligned}(-2a)(3b)(4c) &= (-2)\cdot 3\cdot 4abc\\ &= -24abc.\end{aligned}$$

Similarly,

$$\begin{aligned}(-4x)(5y)(-3c) &= (-4)5(-3)xyc\\ &= 60xyc.\end{aligned}$$

Note that the product of two minus signs gives a plus sign.

Example. We have

$$(-1)(-1) = 1.$$

To see this, all we have to do is apply our rule

$$-(ab) = (-a)b = a(-b).$$

We find

$$(-1)(-1) = -(1(-1)) = -(-1) = 1.$$

Example. More generally, for any integers a, b we have

N10. $$\boxed{(-a)(-b) = ab.}$$

We leave the proof as an exercise. From this we see that a product of two negative numbers is positive, because if a, b are positive and $-a$, $-b$ are therefore negative, then $(-a)(-b)$ is the positive number ab. For instance, -3 and -5 are negative, but

$$(-3)(-5) = -(3(-5)) = -(-(3\cdot 5)) = 15.$$

Example. A product of a negative number and a positive number is negative. For instance, -4 is negative, 7 is positive, and

$$(-4)\cdot 7 = -(4\cdot 7) = -28,$$

so that $(-4)\cdot 7$ is negative.

When we multiply a number with itself several times, it is convenient to use a notation to abbreviate this operation. Thus we write

$$aa = a^2,$$
$$aaa = a^3,$$
$$aaaa = a^4,$$

and in general if n is a positive integer,

$$a^n = aa \cdots a \quad \text{(the product is taken } n \text{ times)}.$$

We say that a^n is the **n-th power** of a. Thus a^2 is the second power of a, and a^5 is the fifth power of a.

If m, n are positive integers, then

N11. $$a^{m+n} = a^m a^n.$$

This simply states that if we take the product of a with itself $m + n$ times, then this amounts to taking the product of a with itself m times and multiplying this with the product of a with itself n times.

Example

$$a^2a^3 = (aa)(aaa) = a^{2+3} = aaaaa = a^5.$$

Example

$$(4x)^2 = 4x \cdot 4x = 4 \cdot 4xx = 16x^2.$$

Example

$$(7x)(2x)(5x) = 7 \cdot 2 \cdot 5xxx = 70x^3.$$

We have another rule for powers, namely

N12. $$(a^m)^n = a^{mn}.$$

This means that if we take the product of a with itself m times, and then take the product of a^m with itself n times, then we obtain the product of a with itself mn times.

Example. We have

$$(a^3)^4 = a^{12}.$$

Example. We have

$$(ab)^n = a^n b^n$$

because

$$\begin{aligned}(ab)^n &= abab \cdots ab \qquad \text{(product of } ab \text{ with itself } n \text{ times)}\\ &= \underbrace{aa \cdots a}_{n}\underbrace{bb \cdots b}_{n}\\ &= a^n b^n.\end{aligned}$$

Example. We have

$$(2a^3)^5 = 2^5(a^3)^5 = 32a^{15}.$$

Example. The population of a city is 300 thousand in 1930, and doubles every 20 years. What will be the population after 60 years?

This is a case of applying powers. After 20 years, the population is $2 \cdot 300$ thousand. After 40 years, the population is $2^2 \cdot 300$ thousand. After 60 years, the population is $2^3 \cdot 300$ thousand, which is a correct answer. Of course, we can also say that the population will be 2 million 400 thousand.

The following three formulas are used constantly. They are so important that they should be thoroughly memorized by reading them out loud and repeating them like a poem, to get an aural memory of them.

$$\boxed{(a + b)^2 = a^2 + 2ab + b^2,} \qquad \boxed{(a - b)^2 = a^2 - 2ab + b^2,}$$

$$\boxed{(a + b)(a - b) = a^2 - b^2.}$$

Proofs. The proofs are carried out by applying repeatedly the rules for multiplication. We have:

$$\begin{aligned}(a + b)^2 = (a + b)(a + b) &= a(a + b) + b(a + b)\\ &= aa + ab + ba + bb\\ &= a^2 + ab + ab + b^2\\ &= a^2 + 2ab + b^2,\end{aligned}$$

which proves the first formula.

$$\begin{aligned}(a - b)^2 = (a - b)(a - b) &= a(a - b) - b(a - b)\\ &= aa - ab - ba + bb\\ &= a^2 - ab - ab + b^2\\ &= a^2 - 2ab + b^2,\end{aligned}$$

which proves the second formula.

$$\begin{aligned}(a + b)(a - b) = a(a - b) + b(a - b) &= aa - ab + ba - bb\\ &= a^2 - ab + ab - b^2\\ &= a^2 - b^2,\end{aligned}$$

which proves the third formula.

Example. We have

$$\begin{aligned}(2 + 3x)^2 &= 2^2 + 2 \cdot 2 \cdot 3x + (3x)^2\\ &= 4 + 12x + 9x^2.\end{aligned}$$

Example. We have

$$\begin{aligned}(3 - 4x)^2 &= 3^2 - 2 \cdot 3 \cdot 4x + (4x)^2\\ &= 9 - 24x + 16x^2.\end{aligned}$$

Example. We have

$$\begin{aligned}(-2a + 5b)^2 &= 4a^2 + 2(-2a)(5b) + 25b^2\\ &= 4a^2 - 20ab + 25b^2.\end{aligned}$$

Example. We have

$$\begin{aligned}(4a - 6)(4a + 6) &= (4a)^2 - 36\\ &= 16a^2 - 36.\end{aligned}$$

We have discussed so far examples of products of two factors. Of course, we can take products of more factors using associativity.

Example. Expand the expression

$$(2x + 1)(x - 2)(x + 5)$$

as a sum of powers of x multiplied by integers.

We first multiply the first two factors, and obtain

$$\begin{aligned}(2x + 1)(x - 2) &= 2x(x - 2) + 1(x - 2)\\ &= 2x^2 - 4x + x - 2\\ &= 2x^2 - 3x - 2.\end{aligned}$$

We now multiply this last expression with $x + 5$ and obtain

$$\begin{aligned}(2x + 1)(x - 2)(x + 5) &= (2x^2 - 3x - 2)(x + 5)\\ &= (2x^2 - 3x - 2)x + (2x^2 - 3x - 2)5\\ &= 2x^3 - 3x^2 - 2x + 10x^2 - 15x - 10\\ &= 2x^3 + 7x^2 - 17x - 10,\end{aligned}$$

which is the desired answer.

EXERCISES

1. Express each of the following expressions in the form $2^m3^na^rb^s$, where m, n, r, s are positive integers.

   a) $8a^2b^3(27a^4)(2^5ab)$
   b) $16b^3a^2(6ab^4)(ab)^3$
   c) $3^2(2ab)^3(16a^2b^5)(24b^2a)$
   d) $24a^3(2ab^2)^3(3ab)^2$
   e) $(3ab)^2(27a^3b)(16ab^5)$
   f) $32a^4b^5a^3b^2(6ab^3)^4$

2. Prove:

$$\begin{aligned}(a + b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3,\\ (a - b)^3 &= a^3 - 3a^2b + 3ab^2 - b^3.\end{aligned}$$

3. Obtain expansions for $(a + b)^4$ and $(a - b)^4$ similar to the expansions for $(a + b)^3$ and $(a - b)^3$ of the preceding exercise.

Expand the following expressions as sums of powers of x multiplied by integers. These are in fact called polynomials. You might want to read, or at least look at, the section on polynomials later in the book (Chapter 13, §2).

4. $(2 - 4x)^2$
5. $(1 - 2x)^2$
6. $(2x + 5)^2$
7. $(x - 1)^2$

8. $(x + 1)(x - 1)$
9. $(2x + 1)(x + 5)$
10. $(x^2 + 1)(x^2 - 1)$
11. $(1 + x^3)(1 - x^3)$
12. $(x^2 + 1)^2$
13. $(x^2 - 1)^2$
14. $(x^2 + 2)^2$
15. $(x^2 - 2)^2$
16. $(x^3 - 4)^2$
17. $(x^3 - 4)(x^3 + 4)$
18. $(2x^2 + 1)(2x^2 - 1)$
19. $(-2 + 3x)(-2 - 3x)$
20. $(x + 1)(2x + 5)(x - 2)$
21. $(2x + 1)(1 - x)(3x + 2)$
22. $(3x - 1)(2x + 1)(x + 4)$
23. $(-1 - x)(-2 + x)(1 - 2x)$
24. $(-4x + 1)(2 - x)(3 + x)$
25. $(1 - x)(1 + x)(2 - x)$
26. $(x - 1)^2(3 - x)$
27. $(1 - x)^2(2 - x)$
28. $(1 - 2x)^2(3 + 4x)$
29. $(2x + 1)^2(2 - 3x)$
30. The population of a city in 1910 was 50,000, and it doubles every 10 years. What will it be (a) in 1970 (b) in 1990 (c) in 2,000?
31. The population of a city in 1905 was 100,000, and it doubles every 25 years. What will it be after (a) 50 years (b) 100 years (c) 150 years?
32. The population of a city was 200 thousand in 1915, and it triples every 50 years. What will be the population

    a) in the year 2215?
    b) in the year 2165?
33. The population of a city was 25,000 in 1870, and it triples every 40 years. What will it be

    a) in 1990?
    b) in 2030?

§4. EVEN AND ODD INTEGERS; DIVISIBILITY

We consider the positive integers 1, 2, 3, 4, 5, . . . , and we shall distinguish between two kinds of integers. We call

$$1, 3, 5, 7, 9, 11, 13, \ldots$$

the **odd integers,** and we call

$$2, 4, 6, 8, 10, 12, 14, \ldots$$

the **even integers.** Thus the odd integers go up by 2 and the even integers go up by 2. The odd integers start with 1, and the even integers start with 2. Another way of describing an even integer is to say that it is a positive integer which can be written in the form $2n$ for some positive integer n. For instance, we can write

$$\begin{aligned} 2 &= 2 \cdot 1, \\ 4 &= 2 \cdot 2, \\ 6 &= 2 \cdot 3, \\ 8 &= 2 \cdot 4, \end{aligned}$$

and so on. Similarly, an odd integer is an integer which differs from an even integer by 1, and thus can be written in the form $2m - 1$ for some positive integer m. For instance,

$$\begin{aligned} 1 &= 2 \cdot 1 - 1, \\ 3 &= 2 \cdot 2 - 1, \\ 5 &= 2 \cdot 3 - 1, \\ 7 &= 2 \cdot 4 - 1, \\ 9 &= 2 \cdot 5 - 1, \end{aligned}$$

and so on. Note that we can also write an odd integer in the form

$$2n + 1$$

if we allow n to be a natural number, i.e., allowing $n = 0$. For instance, we have

$$\begin{aligned} 1 &= 2 \cdot 0 + 1, \\ 3 &= 2 \cdot 1 + 1, \\ 5 &= 2 \cdot 2 + 1, \\ 7 &= 2 \cdot 3 + 1, \\ 9 &= 2 \cdot 4 + 1, \end{aligned}$$

and so on.

Theorem 1. *Let a, b be positive integers.*
If a is even and b is even, then $a + b$ is even.
If a is even and b is odd, then $a + b$ is odd.
If a is odd and b is even, then $a + b$ is odd.
If a is odd and b is odd, then $a + b$ is even.

Proof. We shall prove the second statement, and leave the others as exercises. Assume that a is even and that b is odd. Then we can write

$$a = 2n \qquad \text{and} \qquad b = 2k + 1$$

for some positive integer n and some natural number k. Then

$$\begin{aligned} a + b &= 2n + 2k + 1 \\ &= 2(n + k) + 1 \\ &= 2m + 1 \qquad \text{(letting } m = n + k). \end{aligned}$$

This proves that $a + b$ is odd.

Theorem 2. *Let a be a positive integer. If a is even, then a^2 is even. If a is odd, then a^2 is odd.*

Proof. Assume that a is even. This means that $a = 2n$ for some positive integer n. Then

$$a^2 = 2n \cdot 2n = 2(2n^2) = 2m,$$

where $m = 2n^2$ is a positive integer. Thus a^2 is even.

Next, assume that a is odd, and write $a = 2n + 1$ for some natural number n. Then

$$\begin{aligned} a^2 = (2n + 1)^2 &= (2n)^2 + 2(2n)1 + 1^2 \\ &= 4n^2 + 4n + 1 \\ &= 2(2n^2 + 2n) + 1 \\ &= 2k + 1, \qquad \text{where } k = 2n^2 + 2n. \end{aligned}$$

Hence a^2 is odd, thus proving our theorem.

Corollary. *Let a be a positive integer. If a^2 is even, then a is even. If a^2 is odd, then a is odd.*

Proof. This is really only a reformulation of the theorem, taking into account ordinary logic. If a^2 is even, then a cannot be odd because the square of an odd number is odd. If a^2 is odd, then a cannot be even because the square of an even number is even.

We can generalize the property used to define an even integer. Let d be a positive integer and let n be an integer. We shall say that d **divides** n, or that n **is divisible by** d if we can write

$$n = dk$$

for some integer k. Thus an even integer is a positive integer which is divisible by 2. According to our definition, the number 9 is divisible by 3 because

$$9 = 3 \cdot 3.$$

Also, 15 is divisible by 3 because

$$15 = 3 \cdot 5.$$

Also, -30 is divisible by 5 because

$$-30 = 5(-6).$$

Note that every integer is divisible by 1, because we can always write

$$n = 1 \cdot n.$$

Furthermore, every positive integer is divisible by itself.

EXERCISES

1. Give the proofs for the cases of Theorem 1 which were not proved in the text.

2. Prove: If a is even and b is any positive integer, then ab is even.

3. Prove: If a is even, then a^3 is even.

4. Prove: If a is odd, then a^3 is odd.

5. Prove: If n is even, then $(-1)^n = 1$.

6. Prove: If n is odd, then $(-1)^n = -1$.

7. Prove: If m, n are odd, then the product mn is odd.

Find the largest power of 2 which divides the following integers.

8. 16	9. 24	10. 32	11. 20
12. 50	13. 64	14. 100	15. 36

Find the largest power of 3 which divides the following integers.

16. 30	17. 27	18. 63	19. 99
20. 60	21. 50	22. 42	23. 45

24. Let a, b be integers. Define $a \equiv b \pmod 5$, which we read "a is **congruent to b modulo** 5", to mean that $a - b$ is divisible by 5. Prove: If $a \equiv b \pmod 5$ and $x \equiv y \pmod 5$, then

$$a + x \equiv b + y \pmod 5$$

and

$$ax \equiv by \pmod 5.$$

25. Let d be a positive integer. Let a, b be integers. Define

$$a \equiv b \pmod d$$

to mean that $a - b$ is divisible by d. Prove that if $a \equiv b \pmod d$ and $x \equiv y \pmod d$, then

$$a + x \equiv b + y \pmod d$$

and

$$ax \equiv by \pmod d.$$

26. Assume that every positive integer can be written in one of the forms $3k$, $3k + 1$, $3k + 2$ for some integer k. Show that if the square of a positive integer is divisible by 3, then so is the integer.

§5. RATIONAL NUMBERS

By a **rational number** we shall mean simply an ordinary fraction, that is a quotient

$$\frac{m}{n} \qquad \text{also written} \qquad m/n,$$

where m, n are integers and $n \neq 0$. In taking such a quotient m/n, we emphasize that **we cannot divide by 0,** and thus we must always be sure that $n \neq 0$. For instance,

$$\frac{1}{4}, \frac{2}{3}, -\frac{3}{4}, -\frac{5}{7}$$

are rational numbers. Finite decimals also give us examples of rational numbers. For instance,

$$1.4 = \frac{14}{10} \qquad \text{and} \qquad 1.41 = \frac{141}{100}.$$

Just as we did with the integers, we can represent the rational numbers on the line. For instance, $\frac{1}{2}$ lies one-half of the way between 0 and 1, while

$\frac{2}{3}$ lies two-thirds of the way between 0 and 1, as shown on the following picture.

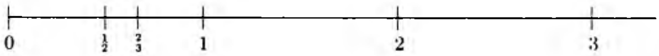


Fig. 1-5

The negative rational number $-\frac{3}{4}$ lies on the opposite side of 0 at a distance $\frac{3}{4}$ from 0. On the next picture, we have drawn $-\frac{3}{4}$ and $-\frac{5}{4}$.

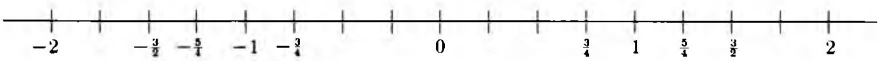


Fig. 1-6

There is no unique representation of a rational number as a quotient of two integers. For instance, we have

$$\frac{1}{2} = \frac{2}{4}.$$

We can interpret this geometrically on the line. If we cut up the segment between 0 and 1 into four equal pieces, and we take two-fourths of them, then this is the same as taking one-half of the segment. Picture:

Fig. 1-7

We need a general rule to determine when two expressions of quotients of integers give the same rational numbers. We assume this rule without proof. It is stated as follows.

Rule for cross-multiplying. *Let m, n, r, s be integers and assume that $n \neq 0$ and $s \neq 0$. Then*

$$\frac{m}{n} = \frac{r}{s} \quad \textit{if and only if} \quad ms = rn.$$

The name "cross-multiplying" comes from our visualization of the rule in the following diagram:

$$\frac{m}{n} \times \frac{r}{s}.$$

Example. We have

$$\frac{1}{2} = \frac{2}{4}$$

because

$$1 \cdot 4 = 2 \cdot 2.$$

Also, we have

$$\frac{3}{7} = \frac{9}{21}$$

because

$$3 \cdot 21 = 9 \cdot 7$$

(both sides are equal to 63).

We shall make no distinction between an integer m and the rational number $m/1$. Thus we write

$$m = m/1 = \frac{m}{1} \cdot$$

With this convention, we see that every integer is also a rational number. For instance, $3 = 3/1$ and $-4 = -4/1$.

Observe the special case of cross-multiplying when one side is an integer. For instance:

$$\frac{2n}{5} = \frac{6}{1}, \qquad \frac{2n}{5} = 6, \qquad 2n = 30, \qquad n = \frac{30}{2} = 15$$

are all equivalent formulations of a relation involving n.

Of course, cross-multiplying also works with negative numbers. For instance,

$$\frac{-4}{5} = \frac{8}{-10}$$

because

$$(-4)(-10) = 8 \cdot 5$$

(both sides are equal to 40).

Remark. For the moment, we are dealing with quotients of integers and describing how they behave. In the next section we shall deal with multiplicative inverses. There, you can see how the rule for cross-multiplication can in fact be proved from properties of such an inverse. Some people view this proof as the reason why cross-multiplication "works". However, in some contexts, one wants to *define* the multiplicative inverse by using the rule for cross-multiplication. This is the reason for emphasizing it here independently.

Cancellation rule for fractions. *Let a be a non-zero integer. Let m, n be integers, $n \neq 0$. Then*

$$\frac{am}{an} = \frac{m}{n} \cdot$$

Proof. To test equality, we apply the rule for cross-multiplying. We must verify that

$$(am)n = m(an),$$

which we see is true by associativity and commutativity.

The examples which we gave are special cases of this cancellation rule. For instance

$$\frac{-4}{5} = \frac{(-2)(-4)}{(-2)5} = \frac{8}{-10}.$$

In dealing with quotients of integers which may be negative, it is useful to observe that

$$\boxed{\frac{-m}{n} = \frac{m}{-n}.}$$

This is proved by cross-multiplying, namely we must verify that

$$(-m)(-n) = mn,$$

which we already know is true.

The cancellation rule leads us to use the notion of divisibility already mentioned in §4. Indeed, suppose that d is a positive integer and m, n are divisible by d (or as we also say, that d is a **common divisor** of m and n). Then we can write

$$m = dr \quad \text{and} \quad n = ds$$

for some integers r and s, so that

$$\frac{m}{n} = \frac{dr}{ds} = \frac{r}{s}.$$

We see that our cancellation rule is applicable.

Example. We have

$$\frac{10}{15} = \frac{2 \cdot 5}{3 \cdot 5} = \frac{2}{3}$$

because 10 and 15 are both divisible by 5.

We say that a rational number is **positive** if it can be written in the form m/n, where m, n are positive integers. Let a be a positive rational number. We shall say that a is **expressed in lowest form** as a fraction

$$a = \frac{r}{s}$$

where r, s are positive integers if the only common divisor of r and s is 1.

Theorem 3. *Any positive rational number has an expression as a fraction in lowest form.*

Proof. First write a given positive rational number as a quotient of positive integers m/n. We know that 1 is a common divisor of m and n. Furthermore, any common divisor is at most equal to m or n. Thus among all common divisors there is a greatest one, which we denote by d. Thus we can write

$$m = dr \quad \text{and} \quad n = ds$$

with positive integers r and s. Our rational number is equal to

$$\frac{m}{n} = \frac{dr}{ds} = \frac{r}{s}.$$

All we have to do now is to show that the only common divisor of r and s is 1. Suppose that e is a common divisor which is greater than 1. Then we can write

$$r = ex \quad \text{and} \quad s = ey$$

with positive integers x and y. Hence

$$m = dr = dex \quad \text{and} \quad n = ds = dey.$$

Therefore de is a common divisor for m and n, and is greater than d since e is greater than 1. This is impossible because we assumed that d was the greatest common divisor of m and n. Therefore 1 is the only common divisor of r and s, and our theorem is proved.

Example. Any positive rational number can be expressed as a quotient m/n, where m, n are positive integers which are not both even, because if m/n is the expression of this rational number in lowest form, then 2 cannot divide both m and n, and therefore at least one of them must be odd.

Let

$$\frac{m}{n} \quad \text{and} \quad \frac{r}{s}$$

be rational numbers, expressed as quotients of integers. We can put these rational numbers over a common denominator ns by writing

$$\frac{m}{n} = \frac{ms}{ns} \quad \text{and} \quad \frac{r}{s} = \frac{nr}{ns}.$$

For instance, to put 3/5 and 5/7 over the common denominator $5 \cdot 7 = 35$, we write

$$\frac{3}{5} = \frac{3 \cdot 7}{5 \cdot 7} = \frac{21}{35} \quad \text{and} \quad \frac{5}{7} = \frac{5 \cdot 5}{7 \cdot 5} = \frac{25}{35}.$$

This leads us to the formula for the addition of rational numbers. Consider first a special case, when the rational numbers have a common denominator, for instance,

$$\frac{3}{5} + \frac{8}{5} = \frac{11}{5}.$$

This is reasonable just from the interpretation of rational numbers: If we have three-fifths of something, and add eight-fifths of that same thing, then we get eleven-fifths of that thing. In general, we can write the rule for addition when the rational numbers have a common denominator as

$$\boxed{\frac{a}{d} + \frac{b}{d} = \frac{a+b}{d}.}$$

Example. We have

$$\frac{-5}{8} + \frac{2}{8} = \frac{-3}{8}.$$

When the rational numbers do not have a common denominator, we get the formula for their addition by putting them over a common denominator. Namely, let $\frac{m}{n}$ and $\frac{r}{s}$ be rational numbers, expressed as quotients of integers m, n and r, s with $n \neq 0$ and $s \neq 0$. Then we have seen that

$$\frac{m}{n} = \frac{sm}{sn} \quad \text{and} \quad \frac{r}{s} = \frac{nr}{ns}.$$

Thus our rational numbers now have the common denominator sn, and thus the formula for addition in this general case is

$$\boxed{\frac{m}{n} + \frac{r}{s} = \frac{ms + rn}{ns}.}$$

Example. We have

$$\frac{3}{5} + \frac{4}{7} = \frac{3 \cdot 7 + 4 \cdot 5}{35} = \frac{21 + 20}{35} = \frac{41}{35}.$$

Example. We have

$$\frac{-5}{2} + \frac{3}{7} = \frac{(-5) \cdot 7 + 2 \cdot 3}{14} = \frac{-29}{14}.$$

Example. We have

$$\frac{3}{-4} + \frac{5}{7} = \frac{21 - 20}{-28} = \frac{1}{-28}.$$

Using our rule for adding rational numbers, we conclude at once:

The sum of positive rational numbers is also positive.

Observe that our number 0 has the property that

$$\boxed{\frac{0}{n} = \frac{0}{1} = 0}$$

for any integer $n \neq 0$. Indeed, applying our test for the equality of two fractions, we must verify that

$$0 \cdot 1 = 0 \cdot n,$$

and this is true because both sides are equal to 0.

For any rational number a, we have

$$\boxed{0 + a = a + 0 = a.}$$

This is easily seen using the analogous property for integers. Namely, write $a = m/n$, where m, n are integers, and $n \neq 0$. Then

$$0 + a = \frac{0}{n} + \frac{m}{n} = \frac{0 + m}{n} = \frac{m}{n} = a,$$

and similarly on the other side.

Let $a = m/n$ be a rational number, where m, n are integers and $n \neq 0$. Then we have

$$\frac{-m}{n} + \frac{m}{n} = \frac{-m + m}{n} = 0.$$

For this reason, we shall write

$$\boxed{\frac{-m}{n} = -\frac{m}{n}.}$$

By a previous remark, we also see that

$$\boxed{-\frac{m}{n} = \frac{m}{-n}.}$$

This shows how a minus sign can be moved around the various terms of a fraction without changing the value of the fraction.

A rational number which can be written as a fraction

$$-\frac{m}{n} = \frac{-m}{n} = \frac{m}{-n}$$

where m, n are positive integers will be called **negative.** For example, the number

$$\frac{3}{-5} = \frac{-3}{5} = -\frac{3}{5}$$

is negative. Using the definition of addition of rational numbers, you can easily verify for yourselves that a sum of negative rational numbers is negative.

Addition of rational numbers satisfies the properties of commutativity and associativity.

Just as we did for integers, the above statement will be accepted without proof. It is in fact a general property of much more general numbers, which will be restated again for these numbers in the next section.

In §2, we proved a number of properties of addition using only commutativity and associativity, together with the rules

$$0 + a = a \quad \textit{and} \quad a + (-a) = 0.$$

These properties therefore remain valid for rational numbers. Similarly, all the exercises of §2 remain valid for rational numbers.

This remark will again be made later whenever we meet a similar situation. For instance, we see as before that

$$\textit{if } a + b = 0, \textit{ then } b = -a.$$

We just add $-a$ to both sides of the equation $a + b = 0$. In words, we can say: To test whether a given rational number is equal to minus another, all we need to verify is that the sum of the numbers is equal to 0.

We shall now give the formula for **multiplication** of rational numbers. This formula is:

$$\boxed{\frac{m}{n} \cdot \frac{r}{s} = \frac{mr}{ns}.}$$

Thus to take the product of two rational numbers, we multiply their numerators and multiply their denominators. More precisely, the numerator of the product is the product of the numerators, and the denominator of the product is the product of the denominators.

Example. We have

$$\frac{3}{5} \cdot \frac{7}{8} = \frac{21}{40}.$$

Also,

$$\frac{2}{7} \cdot \frac{11}{16} = \frac{22}{112}.$$

We can write this last fraction in simpler form, namely

$$\frac{2}{7} \cdot \frac{11}{16} = \frac{2 \cdot 11}{7 \cdot 2 \cdot 8}.$$

We can then cancel 2 and get

$$\frac{2}{7} \cdot \frac{11}{16} = \frac{11}{56}.$$

This shows that sometimes it is best not to carry out a multiplication before looking at the possibility of cancellations.

Example. We have

$$\frac{-4}{5} \cdot \frac{7}{-3} = \frac{(-4)7}{5(-3)} = \frac{-28}{-15} = \frac{28}{15}.$$

Example. Let $a = m/n$ be a rational number expressed as a quotient of integers. Then

$$a^2 = \left(\frac{m}{n}\right)^2 = \frac{m}{n}\frac{m}{n} = \frac{m^2}{n^2}.$$

Similarly,

$$a^3 = \frac{m}{n}\frac{m}{n}\frac{m}{n} = \frac{m^3}{n^3}.$$

In general, for any positive integer k, we have

$$a^k = \left(\frac{m}{n}\right)^k = \frac{m^k}{n^k}.$$

Example. We have

$$\left(\frac{1}{2}\right)^3 = \frac{1}{2^3} = \frac{1}{8}.$$

Also,

$$\left(\frac{3}{5}\right)^4 = \frac{3^4}{5^4} = \frac{81}{525}.$$

Example. A chemical substance disintegrates in such a way that it gets halved every 10 min. If there are 20 grams (g) of the substance present at a given time, how much will be left after 50 min?

This is easily done. At the end of 10 min, we have $\frac{1}{2} \cdot 20$ g left. At the end of 20 min, we have $\frac{1}{2^2} \cdot 20$ g left, and so on; at the end of 50 min, we have

$$\frac{1}{2^5} \cdot 20 = \frac{20}{32}$$

grams left. This is a correct answer. If you want to put the fraction in lowest form, you may do so, and then you get the answer in the form $\frac{5}{8}$ g. You can also put it in approximate decimals, which we don't do here.

We ask: Is there a positive rational number a whose square is 2? The answer is at first not obvious. Such a number would be a square root of 2. Note that $1^2 = 1 \cdot 1 = 1$ and $2^2 = 4$. Thus the square of 1 is smaller than 2 and the square of 2 is bigger than 2. Any positive square root of 2 will therefore lie between 1 and 2 if it exists. We could experiment with various decimals to see whether they yield a square root of 2. For instance, let us try the decimal just in the middle between 1 and 2. We have

$$(1.5)^2 = 2.25,$$

which is bigger than 2. Thus 1.5 is not a square root of 2, and is too big to be one.

We could try more systematically, namely:

$$(1.1)^2 = 1.21 \quad \text{(too small)},$$
$$(1.2)^2 = 1.44 \quad \text{(too small)},$$
$$(1.3)^2 = 1.69 \quad \text{(too small)},$$
$$(1.4)^2 = 1.96 \quad \text{(too small but coming closer)}.$$

We know that 1.5 is too big, and hence we must go to the next decimal place to try out further.

$$(1.41)^2 = 1.9881 \quad \text{(too small)},$$
$$(1.42)^2 = 2.0164 \quad \text{(too big)}.$$

Thus we must go to the next decimal place for further experimentation. We try successively $(1.411)^2$, $(1.412)^2$, $(1.413)^2$, $(1.414)^2$ and find that they are too small. Computing $(1.415)^2$ we see that it is too big. We could keep on going like this. There are several things to be said about our procedure.

(1) It is very systematic, and could be programmed on a computer.

(2) It gives us increasingly good approximations to a square root of 2, namely it gives us rational numbers whose squares come closer and closer to 2.

However, to find a rational number whose square is 2, the procedure is a bummer because of the following theorem.

Theorem 4. *There is no positive rational number whose square is* 2.

Proof. Suppose that such a rational number exists. We can write it in lowest form m/n by Theorem 3. In particular, not both m and n can be even. We have

$$\left(\frac{m}{n}\right)^2 = \frac{m^2}{n^2} = 2.$$

Consequently, we obtain

$$m^2 = 2n^2,$$

and therefore m^2 is even. By the Corollary of Theorem 2 of §4, we conclude that m must be even, and we can therefore write

$$m = 2k$$

for some positive integer k. Thus we obtain

$$m^2 = (2k)^2 = 4k^2 = 2n^2.$$

We can cancel 2 from both sides of the equation

$$4k^2 = 2n^2,$$

and obtain

$$n^2 = 2k^2.$$

This means that n^2 is even, and as before, we conclude that n itself must be even. Thus from our original assumption that $(m/n)^2 = 2$ and m/n is in lowest form, we have obtained the impossible fact that both m, n are even. This means that our original assumption $(m/n)^2 = 2$ cannot be true, and concludes the proof of our theorem.

A number which is not rational is called **irrational.** From Theorem 4, we see that if a positive number a exists such that $a^2 = 2$, then a must be irrational. We shall discuss this further in the next section dealing with real numbers in general.

Multiplication of rational numbers satisfies the same basic rules as multiplication of integers. We state these once more:

For any rational number a we have $1a = a$ and $0a = 0$. Furthermore, multiplication is associative, commutative, and distributive with respect to addition.

As before, we *assume* these as properties of numbers. Moreover, we have the same remark for multiplication that we did for addition. All the properties of §3 which were proved using only the basic ones are therefore also valid for rational numbers. Thus the formulas which we had, like

$$(a + b)^2 = a^2 + 2ab + b^2,$$

are now seen to be valid for rational numbers as well. All the exercises at the end of §3 are valid for rational numbers.

Example. Solve for a in the equation

$$3a - 1 = 7.$$

We add 1 to both sides of the equation, and thus obtain

$$3a = 7 + 1 = 8.$$

We then divide by 3 and get

$$a = \frac{8}{3} \cdot$$

Example. Solve for x in the equation

$$2(x - 3) = 7.$$

To do this, we use distributivity first, and get the equivalent equation

$$2x - 6 = 7.$$

Next we find

$$2x = 7 + 6 = 13,$$

whence

$$x = \frac{13}{2}.$$

Of course we could have given other arguments to find the answer. For instance, we could first get

$$x - 3 = \frac{7}{2},$$

whence

$$x = \frac{7}{2} + 3.$$

This is a perfectly correct answer. However, we can also give the answer in fraction form. We write $3 = \frac{6}{2}$, and find that

$$x = \frac{7}{2} + \frac{6}{2} = \frac{13}{2}.$$

Example. Solve for x in the equation

$$\frac{3x - 7}{2} + 4 = 2x.$$

We multiply both sides of the equation by 2 and obtain

$$3x - 7 + 8 = 4x.$$

We then add $-3x$ to both sides, to get

$$1 = 4x - 3x = x.$$

This solves our problem.

EXERCISES

1. Solve for a in the following equations.

a) $2a = \frac{3}{4}$ b) $\frac{3a}{5} = -7$ c) $\frac{-5a}{2} = \frac{3}{8}$

2. Solve for x in the following equations.

a) $3x - 5 = 0$ b) $-2x + 6 = 1$ c) $-7x = 2$

3. Put the following fractions in lowest form.

a) $\frac{10}{25}$ b) $\frac{3}{9}$ c) $\frac{30}{25}$ d) $\frac{50}{15}$

e) $\frac{45}{9}$ f) $\frac{62}{4}$ g) $\frac{23}{46}$ h) $\frac{16}{40}$

4. Let $a = m/n$ be a rational number expressed as a quotient of integers m, n with $m \neq 0$ and $n \neq 0$. Show that there is a rational number b such that $ab = ba = 1$.

5. Solve for x in the following equations.

a) $2x - 7 = 21$ b) $3(2x - 5) = 7$ c) $(4x - 1)2 = \frac{1}{4}$

d) $-4x + 3 = 5x$ e) $3x - 2 = -5x + 8$ f) $3x + 2 = -3x + 4$

g) $\frac{4x}{3} + 1 = 3x$ h) $-\frac{3x}{2} + \frac{4}{3} = 5x$ i) $\frac{2x - 1}{3} + 4x = 10$

6. Solve for x in the following equations.

a) $2x - \frac{3}{7} = \frac{x}{5} + 1$ b) $\frac{3}{4}x + 5 = -7x$ c) $\frac{-2}{13}x = 3x - 1$

d) $\frac{4x}{3} + \frac{3}{4} = 2x - 5$ e) $\frac{4(1 - 3x)}{7} = 2x - 1$ f) $\frac{2 - x}{3} = \frac{7}{8}x$

7. Let n be a positive integer. By n **factorial,** written $n!$, we mean the product

$$1 \cdot 2 \cdot 3 \cdots n$$

of the first n positive integers. For instance,

$$2! = 2,$$
$$3! = 2 \cdot 3 = 6,$$
$$4! = 2 \cdot 3 \cdot 4 = 24.$$

a) Find the value of $5!$, $6!$, $7!$, and $8!$.

b) Define $0! = 1$. Define the **binomial coefficient**

$$\binom{m}{n} = \frac{m!}{n!(m-n)!}$$

for any natural numbers m, n such that n lies between 0 and m. Compute the binomial coefficients

$$\binom{3}{0}, \binom{3}{1}, \binom{3}{2}, \binom{3}{3}, \binom{4}{0}, \binom{4}{1}, \binom{4}{2}, \binom{4}{3}, \binom{4}{4},$$
$$\binom{5}{0}, \binom{5}{1}, \binom{5}{2}, \binom{5}{3}, \binom{5}{4}, \binom{5}{5}.$$

The binomial coefficient $\binom{m}{n}$ is equal to the number of ways n things can be selected out of m things. You may want to look at the discussion of Chapter 16, §1 at this time to see why this is so.

c) Show that

$$\binom{m}{n} = \binom{m}{m-n}.$$

d) Show that if n is a positive integer at most equal to m, then

$$\binom{m}{n} + \binom{m}{n-1} = \binom{m+1}{n}.$$

8. Prove that there is no positive rational number a such that $a^3 = 2$.
9. Prove that there is no positive rational number a such that $a^4 = 2$.
10. Prove that there is no positive rational number a such that $a^2 = 3$. You may assume that a positive integer can be written in one of the forms $3k$, $3k + 1$, $3k + 2$ for some integer k. Prove that if the square of a positive integer is divisible by 3, then so is the integer. Then use a similar proof as for $\sqrt{2}$.
11. a) Find a positive rational number, expressed as a decimal, whose square approximates 2 up to 3 decimals.

    b) Same question, but with 4 decimals accuracy instead.
12. a) Find a positive rational number, expressed as a decimal, whose square approximates 3 up to 2 decimals.

    b) Same question but with 3 decimals instead.
13. Find a positive rational number, expressed as a decimal, whose square approximates 5 up to

    a) 2 decimals, b) 3 decimals.

14. Find a positive rational number whose cube approximates 2 up to

a) 2 decimals, b) 3 decimals.

15. Find a positive rational number whose cube approximates 3 to

a) 2 decimals, b) 3 decimals.

16. A chemical substance decomposes in such a way that it halves every 3 min. If there are 6 grams (g) of the substance present at the beginning, how much will be left

a) after 3 min? b) after 27 min? c) after 36 min?

17. A chemical substance reacts in such a way that one third of the remaining substances decomposes every 15 min. If there are 15 g of the substance present at the beginning, how much will be left

a) after 30 min? b) after 45 min? c) after 165 min?

18. A substance reacts in water in such a way that one-fourth of the undissolved part dissolves every 10 min. If you put 25 g of the substance in water at a given time, how much will be left after

a) 10 min? b) 30 min? c) 50 min?

19. You are testing the effect of a noxious substance on bacteria. Every 10 min, one-tenth of the bacteria which are still alive are killed. If the population of bacteria starts with 10^6, how many bacteria are left after

a) 10 min? b) 30 min? c) 50 min?

d) Within which period of 10 min will half the bacteria be killed?

e) Within which period of 10 min will 70% of the bacteria be killed?

f) Within which period of 10 min will 80% of the bacteria be killed?

[*Note:* If one-tenth of those alive are killed, then nine-tenths remain.]

20. A chemical pollutant is being emptied in a lake with 50,000 fishes. Every month, one-third of the fish still alive die from this pollutant. How many fish will be alive after

a) 1 month? b) 2 months?

c) 4 months? d) 6 months?

(Give your answer to the nearest 100.)

e) What is the first month when more than half the fish will be dead?

f) During which month will 80% of the fish be dead?

[*Note:* If one-third die, then two thirds remain.]

21. Every 10 years the population of a city is five-fourths of what it was 10 years before. How many years does it take

a) before the population doubles? b) before it triples?

§6. MULTIPLICATIVE INVERSES

Rational numbers satisfy one property which is not satisfied by integers, namely:

If a is a rational number $\neq 0$, then there exists a rational number, denoted by a^{-1}, such that

$$a^{-1}a = aa^{-1} = 1.$$

Indeed, if $a = m/n$ where m, n are integers $\neq 0$, then $a^{-1} = n/m$ because

$$\frac{m}{n}\frac{n}{m} = \frac{mn}{mn} = 1.$$

We call a^{-1} the **multiplicative inverse of** a.

Example. The multiplicative inverse of $\frac{1}{2}$ is $\frac{2}{1}$, or simply 2, because

$$2 \cdot \frac{1}{2} = 1.$$

The multiplicative inverse of $\frac{2}{3}$ is $\frac{3}{2}$. The multiplicative inverse of $-\frac{5}{7}$ is $-\frac{7}{5}$.

Observe that if a and b are rational numbers such that

$$ab = 1,$$

then

$$b = a^{-1}.$$

Proof. We multiply both sides of the relation $ab = 1$ by a^{-1}, and get

$$a^{-1}ab = a^{-1} \cdot 1 = a^{-1}.$$

Using associativity on the left, we find

$$a^{-1}ab = 1b = b,$$

so that we do find $b = a^{-1}$ as desired.

From the existence of an inverse for non-zero rational numbers, we deduce:

If $ab = 0$, then $a = 0$ or $b = 0$.

Proof. Suppose $a \neq 0$. Multiply both sides of the equation $ab = 0$ by a^{-1}. We get:

$$a^{-1}ab = 0a^{-1} = 0.$$

On the other hand, $a^{-1}ab = 1b = b$, so that we find $b = 0$, as desired.

We shall use the same notation as for quotients of integers in taking quotients of rational numbers. We write

$$\frac{a}{b} \quad \text{or} \quad a/b \qquad \text{instead of} \qquad b^{-1}a \quad \text{or} \quad ab^{-1}.$$

Example. Let $a = \frac{3}{4}$ and $b = \frac{5}{7}$. Then

$$\frac{3/4}{5/7} = \frac{3}{4}\left(\frac{5}{7}\right)^{-1} = \frac{3}{4}\frac{7}{5} = \frac{21}{20}.$$

Example. We have

$$\frac{1+\frac{1}{2}}{2-\frac{4}{3}} = \left(1+\frac{1}{2}\right)\cdot\left(2-\frac{4}{3}\right)^{-1}$$

$$= \frac{2+1}{2}\cdot\left(\frac{6-4}{3}\right)^{-1}$$

$$= \frac{3}{2}\left(\frac{2}{3}\right)^{-1} = \frac{3}{2}\frac{3}{2} = \frac{9}{4}.$$

Our rule for cross-multiplication which applied to quotients of integers applies as well when we want to cross-multiply rational numbers. We state it, and prove it using only the basic properties of addition, multiplication, and inverses.

Cross-multiplication. *Let a, b, c, d be rational numbers, and assume that $b \neq 0$ and $d \neq 0$.*

$$\textit{If } \frac{a}{b} = \frac{c}{d}, \textit{ then } ad = bc.$$

$$\textit{If } ad = bc, \textit{ then } \frac{a}{b} = \frac{c}{d}.$$

Proof. Assume that $a/b = c/d$. We can rewrite this relation in the form

$$b^{-1}a = d^{-1}c.$$

Multiply both sides by db (which is the same as bd). We obtain

$$dbb^{-1}a = bdd^{-1}c,$$

so that

$$da = bc$$

because $bb^{-1}a = 1a = a$, and similarly, $dd^{-1}c = 1c = c$.

Conversely, assume that $ad = bc$. Multiply both sides by $b^{-1}d^{-1}$, which is equal to $d^{-1}b^{-1}$. We find:

$$add^{-1}b^{-1} = d^{-1}b^{-1}bc,$$

whence

$$ab^{-1} = d^{-1}c.$$

This means that $a/b = c/d$, as desired.

Example. By cross-multiplying, we have

$$\frac{3}{x-1} = 2$$

if and only if

$$3 = 2(x-1) = 2x - 2,$$

which is equivalent to

$$3 + 2 = 2x.$$

Thus we can solve for x, and get $x = \frac{5}{2}$.

Example. By cross-multiplying we have

$$\frac{4+x}{\frac{1}{2}x} = 5$$

if and only if

$$4 + x = 5 \cdot \frac{1}{2}x = \frac{5x}{2}.$$

Again by cross-multiplication this is equivalent to

$$2(4 + x) = 5x,$$

or

$$8 + 2x = 5x.$$

Subtracting $2x$ from both sides of this equation, we solve for x, and get

$$x = \frac{8}{3}.$$

Cancellation law for multiplication. *Let a be a rational number $\neq 0$.*

If $ab = ac$, then $b = c$.

Proof. Multiply both sides of the equation $ab = ac$ by a^{-1}. We get

$$a^{-1}ab = a^{-1}ac,$$

whence $b = c$.

We also have a **cancellation law** similar to that for quotients of integers.

If a, b, c, d are rational numbers and $a \neq 0$, $c \neq 0$, then

$$\frac{ab}{ac} = \frac{b}{c}.$$

This can be verified, for instance, by cross-multiplication, because we have

$$abc = bac$$

(using commutativity and associativity).

Thus we can operate with fractions formed with rational numbers much as we could operate with fractions formed with integers.

Example. If a/b and c/d are two quotients of rational numbers (and $b \neq 0$, $d \neq 0$), then we can put them over a "common denominator" and write

$$\frac{a}{b} = \frac{ad}{bd}, \qquad \frac{c}{d} = \frac{bc}{bd}.$$

Example. If x, y, b are rational numbers and $b \neq 0$, then we can add quotients in a manner similar to the addition for quotients of integers, namely

$$\begin{aligned} \frac{x}{b} + \frac{y}{b} &= b^{-1}x + b^{-1}y \\ &= b^{-1}(x + y) && \text{by distributivity} \\ &= \frac{x + y}{b} && \text{by definition.} \end{aligned}$$

Combining this with the "common denominator" procedure of the preceding example, we find

$$\boxed{\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.}$$

This formula is entirely analogous to the formula expressing the sum of two rational numbers.

Example. Show that

$$\frac{1}{x-y} + \frac{1}{x+y} = \frac{2x}{x^2-y^2}.$$

To do this, we add the two quotients on the left by our general formula which we just derived, and get:

$$\frac{1(x+y)+1(x-y)}{(x-y)(x+y)} = \frac{x+y+x-y}{x^2-y^2} = \frac{2x}{x^2-y^2},$$

as was to be shown.

Remark. In the preceding example, the quotients $1/(x - y)$ and $1/(x + y)$ make no sense if $x - y = 0$ or $x + y = 0$. In such instances, we assume tacitly that x and y are such that $x - y \neq 0$ and $x + y \neq 0$. In the sequel we shall sometimes omit the explicit mention of such conditions if there is no danger of confusion.

Example. Solve for x in the equation

$$\frac{3x+1}{2x-5} = 4.$$

We cross-multiply. For $2x - 5 \neq 0$, i.e. $x \neq \frac{5}{2}$, we find the equivalent equation

$$3x + 1 = 4(2x - 5) = 8x - 20.$$

Hence

$$8x - 3x = 1 - (-20) = 1 + 20 = 21.$$

This yields finally

$$5x = 21,$$

whence

$$x = \frac{21}{5}.$$

Example. We give an example from the physical world. Suppose that an object is moving along a straight line at constant speed. Let s denote the speed, let d denote the distance traveled by the object, and let t denote the time taken to travel the distance d. Then in physics one verifies the formula

$$d = st.$$

Of course, we must select units of time and distance before we can associate numbers with these. For instance, suppose that the distance traveled is 5 mi, and the time taken is $\frac{1}{2}$ hr. Then the speed is

$$s = d/t = \frac{5 \text{ mi}}{\frac{1}{2} \text{ hr}} = 2 \cdot 5 \text{ mi/hr} = 10 \text{ mi/hr}.$$

Example. A person takes a trip and drives 8 hr, a distance of 400 mi. His average speed is 60 mph on the freeway, and 30 mph when he drives through a town. How long did the person drive through towns during his trip?

To solve this, let x be the length of time the person drives through towns. Then the length of time the person is on the freeway is $8 - x$. The distance driven through towns is therefore equal to $30x$, and the distance driven on freeways is $60(8 - x)$. Since the total distance driven is 400 mi, we have

$$30x + 60(8 - x) = 400.$$

This is equivalent to the equations

$$30x + 480 - 60x = 400$$

and

$$80 = 30x.$$

Thus we find

$$x = \frac{80}{30} = \frac{8}{3}.$$

Hence the person spent $\frac{8}{3}$ hrs driving through towns.

Example. The radiator of a car contains 8 qt of liquid, consisting of water and 40% antifreeze. How much should be drained and replaced by antifreeze if the resultant mixture should have 90% antifreeze?

Let x be the number of quarts which must be drained. After draining this amount, we are left with $(8 - x)$ qt of liquid, of which 40% is antifreeze. Thus we are left with

$$\frac{40}{100}(8 - x) \text{ qt}$$

of antifreeze. Since we now add x qt of antifreeze, we see that x satisfies

$$x + \frac{40}{100}(8 - x) = \frac{90}{100} \cdot 8.$$

From this we can solve for x, transforming this equation into equivalent equations as follows:

$$x + \frac{40}{100} \cdot 8 - \frac{40}{100} x = \frac{90}{100} \cdot 8,$$

which amounts to

$$\frac{60}{100} x = \frac{50}{100} \cdot 8,$$

whence

$$x = \frac{400}{60} = \frac{20}{3}.$$

This is a correct answer, but if you insist on putting the fraction in lowest form, then we can say that $6\frac{2}{3}$ qt should be replaced by antifreeze.

Remark. The above examples, and the exercises, can also be worked using two unknowns. Cf. the end of Chapter 2, §1.

EXERCISES

1. Solve for x in the following equations.

a) $\dfrac{2x - 1}{3x + 2} = 7$

b) $\dfrac{2 - 4x}{x + 1} = \dfrac{3}{4}$

c) $\dfrac{x}{x + 5} = \dfrac{5}{7}$

d) $2x + 5 = \dfrac{3x - 2}{7}$

e) $\dfrac{1 - 2x}{3x + 4} = -3$

f) $\dfrac{-2 - 5x}{-3x - 4} = \dfrac{4}{-3}$

g) $\dfrac{-2 - 7x}{4} + 1 = \dfrac{1 - x}{5}$

h) $\dfrac{3x + 1}{4 - 2x} + \dfrac{7}{3} = 0$

i) $\dfrac{-2 - 4x}{3} = \dfrac{x - 1}{4} + 5$

2. Prove the following relations. It is assumed that all values of x and y which occur are such that the denominators in the indicated fractions are not equal to 0.

a) $\dfrac{1}{x+y} - \dfrac{1}{x-y} = \dfrac{-2y}{x^2-y^2}$ b) $\dfrac{x^3-1}{x-1} = 1 + x + x^2$

c) $\dfrac{x^4-1}{x-1} = 1 + x + x^2 + x^3$

d) $\dfrac{x^n-1}{x-1} = x^{n-1} + x^{n-2} + \cdots + x + 1$. [*Hint:* Cross-multiply and cancel as much as possible.]

3. Prove the following relations.

a) $\dfrac{1}{2x+y} + \dfrac{1}{2x-y} = \dfrac{4x}{4x^2-y^2}$

b) $\dfrac{2x}{x+5} - \dfrac{3x+1}{2x+1} = \dfrac{x^2-14x-5}{2x^2+11x+5}$

c) $\dfrac{1}{x+3y} + \dfrac{1}{x-3y} = \dfrac{2x}{x^2-9y^2}$

d) $\dfrac{1}{3x-2y} + \dfrac{x}{x+y} = \dfrac{x+y+3x^2-2xy}{3x^2+xy-2y^2}$

For more exercises of this type, see Chapter 13, §2

4. Prove the following relations.

a) $\dfrac{x^3-y^3}{x-y} = x^2 + xy + y^2$

b) $\dfrac{x^4-y^4}{x-y} = x^3 + x^2y + xy^2 + y^3$

c) Let

$$x = \frac{1-t^2}{1+t^2} \quad \text{and} \quad y = \frac{2t}{1+t^2}.$$

Show that $x^2 + y^2 = 1$.

5. Prove the following relations.

a) $\dfrac{x^3+1}{x+1} = x^2 - x + 1$

b) $\dfrac{x^5 + 1}{x + 1} = x^4 - x^3 + x^2 - x + 1$

c) If n is an odd integer, prove that

$$\frac{x^n + 1}{x + 1} = x^{n-1} - x^{n-2} + x^{n-3} - \cdots - x + 1.$$

[*Hint:* Cross-multiply.]

6. Assume that a particle moving with uniform speed on a straight line travels a distance of $\frac{5}{4}$ ft at a speed of $\frac{2}{3}$ ft/sec. What time did it take the particle to do that?

7. If a solid has uniform density d, occupies a volume v, and has mass m, then we have the formula

$$m = vd.$$

Find the density if

a) $m = \frac{3}{10}$ lb and $v = \frac{2}{3}$ in^3, b) $m = 6$ lb and $v = \frac{4}{3}$ in^3.

c) Find the volume if the mass is 15 lb and the density is $\frac{2}{3}$ lb/in^3.

8. Let F denote temperature in degrees Fahrenheit, and C the temperature in degrees centigrade. Then F and C are related by the formula

$$C = \tfrac{5}{9}(F - 32).$$

Find C when F is

a) 32, b) 50, c) 99, d) 100, e) -40.

9. Let F and C be as in Exercise 8. Find F when C is:

a) 0, b) -10, c) -40, d) 37, e) 40, f) 100.

10. In electricity theory, one denotes the current by I, the resistance by R, and the voltage by E. These are related by the formula

$$E = IR$$

(with appropriate units). Find the resistance when the voltage and current are:

a) $E = 10$, $I = 3$; b) $E = 220$, $I = 10$.

11. A solution contains 35% alcohol and 65% water. If you start with 12 cm^3 (cubic centimeters) of solution, how much water must be added to make the percentage of alcohol equal to

a) 20%? b) 10%? c) 5%?

2. A plane travels 3,000 mi in 4 hr. When the wind is favorable, the plane averages 900 mph. When the wind is unfavorable, the plane averages 500 mph. During how many hours was the wind favorable?

3. Tickets for a performance sell at $5.00 and $2.00. The total amount collected was $4,100, and there are 1,300 tickets in all. How many tickets of each price were sold?

4. A salt solution contains 10% salt and weighs 80 g. How much pure water must be added so that the percentage of salt drops to

   a) 4%? b) 6%? c) 8%?

5. How many quarts of water must you add to 6 qt of pure alcohol to get a mixture containing

   a) 25% alcohol? b) 20% alcohol? c) 15% alcohol?

6. A boat travels a distance of 500 mi, along two rivers, for 50 hr. The current goes in the same direction as the boat along one river, and then the boat averages 20 mph. The current goes in the opposite direction along the other river, and then the boat averages 8 mph. During how many hours was the boat on the first river?

7. How much water must evaporate from a salt solution weighing 2 lb and containing 25% salt, if the remaining mixture must contain

   a) 40% salt? b) 60% salt?

8. The radiator of a car can contain 10 gal of liquid. If it is half full with a mixture having 60% antifreeze and 40% water, how much more water must be added so that the resulting mixture has only

   a) 40% antifreeze? b) 10% antifreeze?

   Will it fit in the radiator?