كلية العلوم

القسم : **الرياضيات**

السنة : **الثانية**

المادة : معادلات تفاضلية٢

المحاضرة : الثالثة/عملي/

{{ A to Z مكتبة }}

Facebook Group : A to Zمكتبة

1) أوجد حل لقيم x الكبيرة:

$$2x^2(x-1)y'' + x(3x+1)y' - 2y = 0$$

$$x = \infty$$

نفرض أن $t = \frac{1}{x}$

$$dt = \frac{-1}{x^2}dx \Rightarrow \frac{dt}{dx} = -t^2$$

$$y'(x) = -t^2 y'_{(t)} \cancel{(t)}$$

$$y''(x) = \frac{d\left(\frac{-1}{x^2}y'_t\right)}{dx}$$

$$y'_t \frac{d\left(\frac{-1}{x^2}\right)}{dx} + \frac{dy'_t}{dx}\left(\frac{-1}{x^2}\right) =$$

$$y'_t\left(\frac{2}{x^3}\right) + \frac{dy'_t}{dt}\cdot\frac{dt}{dx}\cdot\left(\frac{-1}{x^2}\right) = \frac{2}{x^3}y'_t + y'_t\frac{1}{x^2}$$

$$= \boxed{2t^3 y'_t + t^4 y''_t}$$

نعوض في المعادلة:

$$2\left(\frac{1}{t^2}\right)\left(\frac{1}{t}-1\right)\left(2t^3y'_t + t^4y''_t\right) + \frac{1}{t}\left(\frac{3}{t}+1\right)\left(-t^2y'_t\right) - 2y = 0$$

$$\Rightarrow 2(t-t^2)y''_t + (1-5t)y'_t - 2y = 0 \quad (*)$$

$$\Rightarrow y''_t + \frac{1-5t}{2(t-t^2)}y'_t - \frac{2}{2(t-t^2)}y = 0$$

$$\Rightarrow y''_t + \frac{1-5t}{2(t-t^2)}y'_t - \cancel{\frac{2}{2(t-t^2)}}\frac{1}{(t-t^2)}y = 0$$

$$\Rightarrow P(t) = \frac{1-5t}{2(t-t^2)} \quad \& \quad q(t) = \frac{-1}{t-t^2}$$

$\Leftarrow$ $t=0$ نقطة شاذة للمعادلة لأنها شاذة بالنسبة لـ

$p(x)$ و $q(x)$

لها حل من الشكل:

$$y = \sum_{n=0}^{\infty} C_n t^{n+\alpha}$$

$$y' = \sum_{n=0}^{\infty} (n+\alpha) C_n t^{n+\alpha-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+\alpha-1)(n+\alpha) C_n t^{n+\alpha-2}$$

نعوض في المعادلة $(*)$

$$2\sum_{n=0}^{\infty}(n+\alpha-1)(n+\alpha)C_n t^{n+\alpha-1} - 2\sum_{n=0}^{\infty}(n+\alpha-1)(n+\alpha)C_n t^{n+\alpha} +$$

$$\sum_{n=0}^{\infty}(n+\alpha)C_n t^{n+\alpha-1} - 5\sum_{n=0}(n+\alpha)C_n t^{n+\alpha} - 2\sum_{n=0}^{\infty}C_n t^{n+\alpha} = 0$$

$\Rightarrow$

$$\sum_{n=0}^{\infty}\left[2(n+\alpha-1)(n+\alpha)+(n+\alpha)\right]C_n t^{n+\alpha-1}+\sum_{n=0}^{\infty}\left[-2(n+\alpha-1)(n+\alpha)-5(n+\alpha)-2\right]C_n t^{n+\alpha}=0$$

$$\Rightarrow \sum_{n=0}^{\infty}\left[(n+\alpha)(2n+2\alpha-2+1)\right]C_n t^{n+\alpha-1}+\sum_{n=0}^{\infty}\left[(n+\alpha)(-2n-2\alpha+2-5)-2\right]C_n t^{n+\alpha}=0$$

لعامل المشترك والاختزال نحصل على (*)

$$\Rightarrow \sum_{n=0}^{\infty}\left[(n+\alpha)(2n+2\alpha-1)\right]C_n t^{n+\alpha-1}+\sum_{n=0}^{\infty}\left[-2(n+\alpha)^2-3(n+\alpha)-2\right]C_n t^{n+\alpha}=0$$

(*)

الآن

$$(*)=\sum_{n=0}^{\infty}\left[(n+\alpha)(2n+2\alpha-1)\right]C_n t^{n+\alpha-1}$$

نضع $n=n+1$

$$\sum_{n+1=0}^{\infty}\left[(n+1+\alpha)(2n+2\alpha+1)\right]C_{n+1} t^{n+\alpha}=$$

$$\sum_{n=-1}^{\infty}\left[(n+\alpha+1)(2n+2\alpha+1)\right]C_{n+1} t^{n+\alpha}$$

$$\alpha(2\alpha-1)C_0 t^{\alpha-1}+\sum_{n=0}^{\infty}\left[(n+\alpha+1)(2n+2\alpha+1)C_{n+1} t^{n+\alpha}\right]$$

$$\Rightarrow$$

$$\alpha(2\alpha-1)C_0 t^{\alpha-1}+\sum_{n=0}^{\infty}\left[\left(-2(n+\alpha)^2-3(n+\alpha)-2\right)C_n t^{n+\alpha}+(n+\alpha+1)(2n+2\alpha+1)C_{n+1} t^{n+\alpha}\right]=0$$

$$\alpha(2\alpha-1)C_0=0 \qquad \because C_0\neq 0$$

$$\Rightarrow \alpha(2\alpha-1)=0 \Rightarrow$$

$$\alpha_1=0 \;,\; \alpha_2=\frac{1}{2}$$

$$\alpha_1-\alpha_2=\frac{-1}{2}$$

الفرق عدد غير صحيح

نجد الدستور التدريجي ونعوض α_1 في الدستور التدريجي نحصل على y_1 ونعوض α_2 فنحصل على y_2

$$\Rightarrow \boxed{y=Ay_1+By_2}$$

$$\left(-2(n+\alpha)^2-3(n+\alpha)-2\right)C_n=-(n+\alpha+1)(2n+2\alpha+1)C_{n+1}$$

$$\Rightarrow$$

$$C_{n+1}=\frac{-2(n+\alpha)^2-3(n+\alpha)-2}{-(n+\alpha+1)(2n+2\alpha+1)}C_n$$

نعوض $\alpha_1=0$

$$\Rightarrow C_{n+1}=\frac{-2n^2-3n-2}{-(n+1)(2n+1)}C_n$$

$$\boxed{C_0=1}$$

$$C_1=\frac{0-0-2}{-1(1)}C_0=2C_0=2 \Rightarrow \boxed{C_1=2}$$

$$C_2=\frac{-2-3-2}{-(2)(3)}C_1=\frac{-7}{6}(2)=\frac{14}{6}=\frac{7}{3} \Rightarrow \boxed{C_2=\frac{7}{3}}$$

$$C_3=\frac{-2(4)-3(2)-2}{-(2+1)(4+1)}C_2=\frac{-8-6-2}{-3(5)}C_2=\boxed{\frac{112}{45}}$$

$$\Rightarrow y_1 = t^{\alpha_1}\left[1 + 2t + \frac{7}{3}t^2 + \frac{112}{45}t^3 + \cdots\right]$$

$$y_1 = \left(\frac{1}{x}\right)^0\left[1 + 2\frac{1}{x} + \frac{7}{3}\left(\frac{1}{x}\right)^2 + \frac{112}{45}\left(\frac{1}{x}\right)^3 + \cdots\right]$$

$$\boxed{y_1 = \left[1 + 2x^{-1} + \frac{7}{3}x^{-2} + \frac{112}{45}x^{-3} + \cdots\right]}$$

نعوض $\alpha_2 = \frac{1}{2}$ في الدستور التدريجي:

$$\Rightarrow C_{n+1} = \frac{-2\left(n+\frac{1}{2}\right)^2 - 3\left(n+\frac{1}{2}\right) - 2}{-\left(n+\frac{3}{2}\right)(2n+2)}C_n$$

$$\boxed{C_0 = 1}$$

$$C_1 = \frac{-2\left(0+\frac{1}{2}\right)^2 - 3\left(0+\frac{1}{2}\right) - 2}{-\left(0+\frac{3}{2}\right)(0+2)}C_0 = \boxed{\frac{4}{3}}$$

$$\boxed{C_2 = \frac{22}{15}} \quad \& \quad \boxed{C_3 = \frac{484}{315}}$$

$\Rightarrow$

$$y_2 = t^{\alpha_2}\left[1 + \frac{4}{3}t + \frac{22}{15}t^2 + \frac{484}{315}t^3 + \cdots\right]$$

$$y_2 = \left(\frac{1}{x}\right)^{\frac{1}{2}}\left[1 + \frac{4}{3}\left(\frac{1}{x}\right) + \frac{22}{15}\left(\frac{1}{x}\right)^2 + \frac{484}{315}\left(\frac{1}{x}\right)^3 + \cdots\right]$$

$$\boxed{y_2 = x^{-\frac{1}{2}}\left[1 + \frac{4}{3}x^{-1} + \frac{22}{15}x^{-2} + \frac{484}{315}x^{-3} + \cdots\right]}$$

$$\Rightarrow y = Ay_1 + By_2$$

$\Rightarrow$

$$y = A\left[1 + 2x^{-1} + \frac{7}{3}x^{-2} + \frac{112}{45}x^{-3} + \dots\right] +$$

$$B x^{-\frac{1}{2}}\left[1 + \frac{4}{3}x^{-1} + \frac{22}{15}x^{-2} + \frac{484}{315}x^{-3} + \dots\right]$$

أوجد الحل العام للجمل الآتية:

① $\underset{(1)}{\frac{dx}{z}} = \underset{(2)}{\frac{dy}{z}} = \underset{(3)}{\frac{dz}{x-y}}$

من (1) و (2) :

$$\frac{dx}{z} = \frac{dy}{z} \Rightarrow dx = dy \Rightarrow x = y + C_1 \Rightarrow$$

$$\boxed{\varphi_1 : x - y = C_1}$$

من 1 و 3 :

$$\frac{dx}{z} = \frac{dz}{x-y}$$

من φ_1 : $x - y = C_1$

$$\Rightarrow \frac{dx}{z} = \frac{dz}{C_1} \Rightarrow C_1\, dx = z\, dz$$

$$C_1 x = \frac{z^2}{2} + C_2$$

$$\Rightarrow C_1 x - \frac{1}{2}z^2 = C_2$$

$$\Rightarrow \boxed{\varphi_2 : (x-y)x - \frac{1}{2}z^2 = C_2}$$

$$\frac{D(\varphi_1, \varphi_2)}{D(z, y)} = \begin{vmatrix} -1 & 0 \\ -x & -z \end{vmatrix} = z \neq 0$$

$\Leftarrow$ φ_1 وَ φ_2 مستقلان خطياً وصيغة التكامل العام هي:

$$\boxed{\begin{aligned} &\varphi_1 : x - y = C_1 \\ &\varphi_2 : (x-y)x - \frac{1}{2}z^2 = C_2 \end{aligned}}$$

(2)

$$\underset{(1)}{\frac{dx}{1}} = \underset{(2)}{\frac{dy}{-2}} = \underset{(3)}{\frac{dz}{3x^2 \sin(y+2x)}}$$

من ① و ②

$$\frac{dx}{1} = \frac{dy}{-2}$$

$$-2dx = dy \Rightarrow 2x + y = C_1$$

$$\Rightarrow \boxed{\varphi_1 : 2x + y = C_1}$$

من ①, ③

$$\frac{dx}{1} = \frac{dz}{3x^2 \sin(y+2x)}$$

من φ_1 نجد أن $C_1 = y + 2x$

$$\Rightarrow dx = \frac{dz}{3x^2 \sin C_1}$$

$$3x^2 \sin C_1 \, dx = dz$$

$$x^3 \sin(2x+y) - z = C_2$$

$$\Rightarrow$$

$$\boxed{\varphi_2 : x^3 \sin(2x+y) - z = C_2}$$

$$\frac{D(\varphi_1, \varphi_2)}{D(y, z)} = \begin{vmatrix} 1 & 0 \\ x^3 \cos(2x+y) & -1 \end{vmatrix} = 0 + 1 = 1 \neq 0$$

⇐ φ_1 و φ_2 مستقلات خطياً و صيغة التكامل العام هي :

$$\boxed{\begin{aligned} &\varphi_1 : 2x + y = C_1 \\ &\varphi_2 : x^3 \sin(2x+y) - z = C_2 \end{aligned}}$$

3

$$\underset{(1)}{\frac{dx}{2z - 3y}} = \underset{(2)}{\frac{dy}{3x - z}} = \underset{(3)}{\frac{dz}{y - 2x}}$$

$$(1) + 2(2) = 3$$

$$\frac{dx + 2dy}{2z - 3y + 6x - 2z} = \frac{dz}{y - 2x}$$

$$\frac{dx + 2dy}{-3\cancel{(y - 2x)}} = \frac{dz}{\cancel{(y - 2x)}} \Rightarrow$$

$$\boxed{\varphi_1 : \frac{-1}{3} x - \frac{2}{3} y - z = C_1}$$

$$x(1) + y(2) + z(3) =$$

$$\frac{x\,dx + y\,dy + z\,dz}{\cancel{2xz} - \cancel{3xy} + \cancel{3xy} - \cancel{yz} + \cancel{zy} - \cancel{2xz}}$$

المقام يصبح يساوي الصفر $\Leftarrow$

$$x\,dx + y\,dy + z\,dz = 0$$

$$\frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2 = C_2$$

$$\boxed{\Rightarrow \varphi_2 : \ \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2 = C_2}$$

$$\frac{D(\varphi_1, \varphi_2)}{D(x, z)} = \begin{vmatrix} x & z \\ \frac{-1}{3} & -1 \end{vmatrix} = x + \frac{1}{3}z \neq 0$$

$\Leftarrow \varphi_1$, φ_2 مستقلات خطياً وصيغة التكامل العام

$\Leftarrow$ ~~φ~~ هي:

$$\varphi_1 : \ \frac{-1}{3}x - \frac{2}{3}y - z = C_1$$

$$\varphi_2 : \ \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2 = C_2$$

انتهت المحاضرة