

حلّ الوظيفة السادسة

السؤال الأول: أنجز التكامل

$$I = \int \frac{dx}{1 + \sin x}$$

الحل: نفرض $t = \tan \frac{x}{2}$

$$\sin x = \frac{2t}{1+t^2}, \quad dx = \frac{2dt}{1+t^2}$$

نعوض نجد:

$$I = \int \frac{dx}{1 + \sin x} = \int \frac{\frac{2}{1+t^2}}{1 + \frac{2t}{1+t^2}} dt = \int \frac{2dt}{(t+1)^2} = -2 \times \frac{1}{t+1} + c = -\frac{2}{\tan \frac{x}{2} + 1} + c$$

السؤال الثاني: أنجز التكامل

$$I = \int \frac{\sin^5 x}{\cos^5 x} dx$$

الحل:

$$f(\sin x, \cos x) = \frac{\sin^5 x}{\cos^5 x}$$

$$f(-\sin x, -\cos x) = f(\sin x, \cos x)$$

نفرض $t = \tan x$ عندئذ $dt = \frac{1}{\cos^2 x} dx$

$$dt = (1 + \tan^2 x) dx = (1 + t^2) dx \Rightarrow dx = \frac{dt}{1+t^2}$$

$$I = \int \frac{\sin^5 x}{\cos^5 x} dx = \int \frac{t^5}{(t^2+1)} dt = \int \left(t^3 - t + \frac{t}{t^2+1} \right) dt = \frac{t^4}{4} - \frac{t^2}{2} + \frac{1}{2} \ln|t^2+1| + c$$

$$= \frac{t^4}{4} - \frac{t^2}{2} + \frac{1}{2} \ln(t^2+1) + c = \frac{(\tan x)^4}{4} - \frac{(\tan x)^2}{2} + \frac{1}{2} \ln(\tan^2 x + 1) + c$$

السؤال الثالث: أنجز التكامل التالي

$$I = \int \cos^4 x \sin^2 x dx$$

الحل:

$$I = \int \left(\frac{1 + \cos 2x}{2} \right)^2 \left(\frac{1 - \cos 2x}{2} \right) dx = \frac{1}{8} \int (1 + 2\cos 2x + \cos^2 2x)(1 - \cos 2x) dx$$

$$= \frac{1}{8} \int (1 + \cos 2x - \cos^2 2x - \cos^3 2x) dx$$

$$= \frac{1}{8} \int \left(1 + \cos 2x - \left(\frac{1 + \cos 4x}{2} \right) - \cos 2x(1 - \sin^2 2x) \right) dx$$

$$= \frac{1}{8} \int \left(\frac{1}{2} - \frac{1}{2} \cos 4x + \cos 2x \sin^2 2x \right) dx$$

$$= \int \left(\frac{1}{16} - \frac{1}{16} \cos 4x + \frac{1}{8} \cos 2x \sin^2 2x \right) dx$$

$$= \frac{1}{16} x - \frac{1}{64} \sin 4x + \frac{1}{16} \times \frac{1}{3} \sin^3 2x + c = \frac{1}{16} x - \frac{1}{64} \sin 4x + \frac{1}{58} \sin^3 2x + c$$

السؤال الرابع: أنجز التكامل

$$I = \int \frac{shx + chx}{sh^2 x} dx$$

الحل: نفرض $t = th \frac{x}{2}$

$$shx = \frac{2t}{1-t^2}, \quad chx = \frac{1+t^2}{1-t^2}, \quad dx = \frac{2dt}{1-t^2}$$

نعوض نجد:

$$\begin{aligned} \int \frac{shx + chx}{sh^2 x} dx &= \frac{1}{2} \int \left(\frac{2t + 1 + t^2}{t^2} \right) dt = \frac{1}{2} \int \left(\frac{2}{t} + \frac{1}{t^2} + 1 \right) dt = \ln|t| - \frac{1}{2t} + \frac{1}{2}t + c \\ &= \ln \left| th \frac{x}{2} \right| - \frac{1}{2th \frac{x}{2}} + \frac{1}{2} th \frac{x}{2} + c \end{aligned}$$

السؤال الخامس: أنجز التكامل

$$I = \int \frac{chx}{1 + sh^2 x} dx$$

الحل:

$$\begin{aligned} f(shx, chx) &= \frac{chx}{1 + sh^2 x} \\ f(shx, -chx) &= -f(shx, chx) \end{aligned}$$

نفرض $t = shx$ عندئذ $dt = chx dx$

$$I = \int \frac{chx}{1 + sh^2 x} dx = \int \frac{dt}{1 + t^2} = \arctan(t) + c = \arctan(shx) + c$$