كلية العلوم

القسم : **الرياضيات**

السنة : **الثانية**

المادة : معادلات تفاضلية٢

المحاضرة : الثانية/عملي/

{{ A to Z مكتبة }}

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حل المعادلات التفاضلية:

$\boxed{1}$ $(1+x^2)y'' + xy' - y = 0$

في جوار $x=0$

$p(x) = \frac{x}{1+x^2}$, $q(x) = \frac{1}{x^2+1}$

$x=0$ نقطة عادية لذلك نفرض الحل:

$$y = \sum_{n=0}^{\infty} C_n x^n$$

$$y' = \sum_{n=0}^{\infty} C_n . n . x^{n-1} \quad , \quad y'' = \sum_{n=0}^{\infty} C_n . n . (n-1) x^{n-2}$$

$\Rightarrow$

$$\sum_{n=0}^{\infty} C_n n (n-1) x^{n-2} + \sum_{n=0}^{\infty} C_n . n . (n-1) x^n + \sum_{n=0}^{\infty} C_n . n . x^n - \sum_{n=0}^{\infty} C_n x^n = 0$$

نوحد القوى:

$$\sum_{n=2} C_n . n (n-1) x^{n-2} + \sum_{n=2} C_{n-2} (n-2)(n-3) x^{-2} +$$

$$\sum_{n=2}^{\infty} C_{n-2} (n-2) x^{n-2} - \sum_{n=2} C_{n-2} x^{n-2} = 0$$

$$\sum_{n=2}^{\infty} \left[C_n . n . (n-1) + C_{n-2} \left((n-2)(n-3) + (n-2) - 1 \right) \right] x^{n-2} = 0$$

$$C_n . n (n-1) + C_{n-2} (n^2 - 4n + 3) = 0$$

$$C_n . n (n-1) = -(n-3)(n-1) C_{n-2} \Rightarrow \boxed{C_n = \frac{3-n}{n} C_{n-2}} \quad n \geq 2$$

إيجاد الثوابت:

$$C_2 = \frac{1}{2} C_0$$

$$C_3 = 0 \Rightarrow C_5 = C_7 = \dots = C_{2n+1} = 0$$

$$C_4 = \frac{-1}{4} C_2 = \frac{-1}{8} C_0$$

$$C_6 = \frac{-1}{2} C_4 = \frac{1}{16} C_0$$

$$y = \sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + \frac{1}{2} C_0 x^2 - \frac{1}{8} C_0 x^4 + \frac{1}{16} C_0 x^6 + \dots$$

$$= C_0 \left[1 + \frac{1}{2} x^2 - \frac{1}{8} x^4 + \frac{1}{16} x^6 + \dots \right] + C_1 x$$

$\boxed{2}$ $y'' + (x-1) y' + y = 0$

في جوار $x = 2$

الحل نفرض $t = x - 2$

$$y'' + (t+1) y' + y = 0$$

$x = 2 \Leftarrow t = 0$ نقطة عادية

لذلك نفرض الحل:

$$y = \sum_{n=0}^{\infty} C_n t^n$$

$$y' = \sum_{n=0}^{\infty} n . C_n . t^{n-1} \quad , \quad y'' = \sum_{n=0}^{\infty} n(n-1) C_n t^{n-2}$$

نعوض:

$$\sum_{n=0}^{\infty} n(n-1) C_n t^{n-2} + \sum_{n=0}^{\infty} C_n . n . t^n + \sum_{n=0}^{\infty} C_n . n . t^{n-1} + \sum_{n=0}^{\infty} C_n . t^n = 0$$

$$\sum_{n=2} n(n-1)C_n t^{n-2} + \sum_{n=0} C_n n t^n + \sum_{n=1} C_n n t^{n-1} + \sum_{n=0}^{\infty} C_n t^n = 0$$

$(n \to n+2)$ $(n \to n+1)$

$$\sum_{n=0} (n+2)(n+1)C_{n+2} t^n + \sum_{n=0} C_n n t^n + \sum_{n=0} C_{n+1}(n+1)t^n + \sum_{n=0} C_n t^n = 0$$

$$(n+2)(n+1)C_{n+2} + C_n(n+1) + C_{n+1}(n+1) = 0$$

$$(n+2)(n+1)C_{n+2} = -(n+1)(C_n + C_{n+1})$$

$$C_{n+2} = \frac{-1}{n+2}(C_n + C_{n+1})$$

$$C_2 = \frac{-1}{2}(C_0 + C_1)$$

$$C_3 = \frac{-1}{5}(C_1 + C_2) = \frac{-1}{5}C_1 + \frac{1}{10}C_0 + \frac{1}{10}C_1 = \frac{1}{10}C_0 - \frac{1}{10}C_1$$

$$C_4 = \frac{-1}{6}(C_2 + C_3) = \frac{1}{12}C_0 + \frac{1}{12}C_1 - \frac{1}{60}C_0 + \frac{1}{60}C_1 = \frac{4}{60}C_0 + \frac{6}{60}C_1$$

$$\Rightarrow y = \sum C_n t^n = \sum C_n (x-2)^n$$

$$y = C_0 + C_1(x-2) - \frac{1}{2}(C_0 + C_1)(x-2)^2 + \frac{1}{10}(C_0 - C_1)(x-2)^3 + \dots$$

$$y = C_0\left[1 - \frac{1}{2}(x-2)^2 + \frac{1}{10}(x-2)^3 + \dots\right] + C_1\left[(x-2) - \frac{1}{2}(x-2)^2 - \dots\right]$$

$\boxed{3}$ $2xy'' + (x+1)y' + 3y = 0$

في جوار $x=0$

$p(x) = \dfrac{x+1}{2x}$ لـ نقطة شاذة $x=0$

نعوض:

$$y = \sum_{n=0}^{\infty} C_n x^{n+\alpha}$$

$$y' = \sum_{n=0}^{\infty} C_n (n+\alpha)\, x^{n+\alpha-1}$$

$$y'' = \sum_{n=0}^{\infty} C_n (n+\alpha)(n+\alpha-1)\, x^{n+\alpha-2}$$

نعوض

$$2\sum_{n=0}^{\infty} C_n (n+\alpha)(n+\alpha-1)\, x^{n+\alpha-1} + \sum_{n=0}^{\infty} C_n (n+\alpha)\, x^{n+\alpha} +$$

$$\sum_{n=0}^{\infty} C_n (n+\alpha)\, x^{n+\alpha-1} + \cancel{\sum} \; 3\sum_{n=0}^{\infty} C_n \cdot x^{n+\alpha} = 0$$

$$\sum_{n=0} C_n \left[(n+\alpha)(2n+2\alpha-1)\, x^{n+\alpha-1} \right] + \sum_{\substack{n=0 \\ n \to n-1}}^{\infty} C_n (n+\alpha+3)\, x^{n+\alpha} = 0$$

$$\sum_{n=0}^{\infty} C_n \left[(n+\alpha)(2n+2\alpha-1)\, x^{n+\alpha-1} \right] + \sum_{n=1} C_{n-1} (n+\alpha+2)\, x^{n+\alpha-1} = 0$$

$\boxed{n=0} \Rightarrow$

$$\left. \begin{array}{l} C_0 \cdot \alpha(2\alpha-1) = 0 \rightarrow \alpha = 0 \\ \quad \hookrightarrow \alpha = \frac{1}{2} \end{array} \right\} \text{الفرق بينهما}$$

$\boxed{n \geq 1} \Rightarrow C_n (n+\alpha)(2n+2\alpha-1) + C_{n-1}(n+\alpha+2) = 0$

$$C_n = -\frac{n+\alpha+2}{(n+\alpha)(2n+2\alpha-1)}\, C_{n-1}$$

$$\alpha = 0 \Rightarrow C_n = \frac{-n+2}{n(2n-1)}\, C_{n-1}$$

بالتعويض:

$C_1 = -3\,C_0$

$C_2 = 2\,C_0$

$C_3 = \frac{-2}{3}\,C_0$

$$\Rightarrow y_1 = \sum_{n=0} C_n x^{n+\alpha}$$

$$= C_0 - 3C_0 x + 2C_0 x^2 - \frac{2}{3} C_0 x^3 + \dots$$

$$= C_0 \left[1 - 3x + 2x^2 - \frac{2}{3} x^3 + \dots\right]$$

$$\alpha = \frac{1}{2} \Rightarrow C_n = \frac{-n + \frac{5}{2}}{(n+\frac{1}{2})\,2n} C_{n-1} = \frac{-2n+5}{2n(2n-1)} C_{n-1}$$

$C_1 = \frac{-7}{6} C_0$

$C_2 = \frac{21}{40} C_0$

$C_3 = \frac{-11}{80} C_0$

$$y_2 = \sum C_n x^{n+\frac{1}{2}}$$

$$= x^{\frac{1}{2}} \left[C_0 - \frac{7}{6} C_0 x + \frac{21}{40} C_0 x^2 + \frac{11}{80} C_0 x^3 + \dots\right]$$

$$= x^{\frac{1}{2}} C_0 \left[1 - \frac{7}{6} x + \frac{21}{40} x^2 + \frac{11}{80} x^3 + \dots\right]$$

$$y = A y_1 + B y_2$$

$$x^2 y'' + 3xy' + (1-2x)y = 0 \qquad \boxed{4}$$

في جوار $x = 0$

$p(x) = \frac{3}{x}$ نقطة شاذة $x=0$

نفرض $y = \sum_{n=0}^{\infty} C_n x^{n+\alpha}$

$y' = \sum_{n=0}^{\infty} C_n (n+\alpha) x^{n+\alpha-1}$

$y'' = \sum_{n=0}^{\infty} C_n (n+\alpha)(n+\alpha-1) x^{n+\alpha-2}$

$$\sum_{n=0} C_n (n+\alpha)(n+\alpha-1) x^{n+\alpha} + 3\sum_{n=0} C_n (n+\alpha) x^{n+\alpha} +$$

$$\sum_{n=0} C_n x^{n+\alpha} - 2 \underbrace{\sum_{n=0} C_n x^{n+\alpha+1}}_{n \to n-1} = 0$$

$$\sum_{n=0} C_n \left[(n+\alpha)(n+\alpha-1) + 3(n+\alpha) + 1\right] x^{n+\alpha} - 2\sum_{n=1} C_{n-1} x^{n+\alpha} = 0$$

$n=0 \Rightarrow \alpha(\alpha-1) + 3\alpha + 1 = 0$

$\alpha^2 + 2\alpha + 1 = 0$

$\alpha_1 = \alpha_2 = -1$

جذران متساويان الفرق بينهما صفر

$C_n (n+\alpha+1)^2 - 2C_{n-1} = 0$

$C_n = \frac{2}{(n+\alpha+1)^2}$

$C_1 = \frac{2}{(\alpha+2)^2} C_0$

$C_2 = \frac{2}{(\alpha+3)^2} \cdot \frac{2}{(\alpha+2)^2} C_0 = \frac{2^2}{(\alpha+3)^2(\alpha+2)^2} C_0$

$$C_3 = \frac{2^3}{(\alpha+4)^2(\alpha+3)^2(\alpha+2)^2} C_0$$

نضع $C_0 = 1$

$$y(x,\alpha) = x^{\alpha}\left[1 + \frac{2}{(\alpha+2)^2}x + \frac{2^2}{(\alpha+3)^2(\alpha+2)^2}x^2 + \cdots\right]$$

$$y_1 = y(x,-1) = x^{-1}\left[1 + 2x + x^2 + \cdots\right]$$

$$y_2 = \frac{\partial y(x,\alpha)}{\partial x}\Big|_{\alpha=-1}$$

$$\frac{\partial y(x,\alpha)}{\partial x} = x^{\alpha} \ln x\left[1 + \frac{2}{(\alpha+2)^2}x + \cdots\right] + x^{\alpha}\left[\frac{-4}{(\alpha+2)^3}\right]x - 4\left[\frac{2^2}{(\alpha+2)^3(\alpha+3)^2} + \frac{2^2}{(\alpha+2)^2(\alpha+3)^3}\right]x^2 + \cdots$$

$$y_2 = \frac{\partial y(x,\alpha)}{\partial x}\Big|_{\alpha=-1} = x^{-1}\ln\left[1 + 2x + \cdots\right] + x^{-1}\left[-4x - 6x^2 + \cdots\right]$$

$$y = Ay_1 + By_2$$

$xy'' - 3y' + xy$ $\boxed{5}$

في جوار $x = 0$

$P(x) = \frac{-3}{x}$ ـا شاذة نقطة $x = 0$

نفرض

$$y = \sum_{n=0}^{\infty} C_n x^{n+\alpha}$$

$$y' = \sum_{n=0} C_n (n+\alpha) x^{n+\alpha-1}$$

$$y'' = \sum_{n=0} C_n (n+\alpha)(n+\alpha-1) x^{n+\alpha-2}$$

$$\sum_{n=0}^{\infty} C_n (n+\alpha)(n+\alpha-1) x^{n+\alpha-1} - 3\sum_{n=0} C_n (n+\alpha) x^{n+\alpha-1} +$$

$$\sum_{n=0} C_n x^{n+\alpha+1} = 0 \qquad \boxed{n \to n-2}$$

** $$\sum_{n=0} C_n (n+\alpha)(n+\alpha-4) x^{n+\alpha-1} + \sum_{n=2} C_{n-2} x^{n+\alpha-1} = 0$$

$\boxed{n=0} \Rightarrow \alpha(\alpha-4) = 0 \to \alpha = 0,\ \alpha = 4$ الفرق بينهما عدد صحيح

نفرض $C_1 = 0$

$$C_n = \frac{-1}{(n+\alpha)(n+\alpha-4)} C_{n-2} \qquad n \geq 2$$

$$C_2 = \frac{-1}{(\alpha+2)(\alpha-2)} C_0$$

$$C_3 = \frac{-1}{(\alpha+2)(\alpha-1)} C_1 = 0 = C_7 = C_{2n+1}$$

$$C_4 = \frac{-1}{(\alpha+3)(\alpha-1)} C_2 = \frac{1}{(\alpha-2)(\alpha+2)\alpha(\alpha+4)} C_0$$

خطوة سابقة: نكتب ** نفك السلسلة

$$C_0\alpha(\alpha-4)x^{\alpha-1}+C_1(1+\alpha)(\alpha-3)x^{\alpha}+\sum_{n=2}\left[(n+\alpha)(n+\alpha-1)C_n+C_{n-2}\right]=0$$

$$C_0(\alpha)(\alpha-4)=0$$

$$C_0\neq 0 \rightarrow \alpha=0$$
$$\hookrightarrow \alpha=4$$

$$C_1(1+\alpha)(\alpha-3)=0 \Rightarrow C_1=0$$

من أجل $\alpha=0$ أو $\alpha=4$ غير معدوم

$$\Rightarrow y(x,\alpha)=\sum C_n x^{n+\alpha}$$
$$=x^{\alpha}\left[C_0-\frac{1}{(\alpha+2)(\alpha-2)}C_0x^2+\frac{1}{(\alpha-2)(\alpha+2)\alpha(\alpha+4)}C_0x^4+\cdots\right]$$

من أجل $\alpha=0$ المعاملات غير معرفة

نعوض $C_0=\alpha-0=\alpha$

$$\Rightarrow y(x,\alpha)=x^{\alpha}\left[\alpha-\frac{\alpha}{(\alpha+2)(\alpha-2)}x^2+\frac{1}{(\alpha-2)(\alpha+2)(\alpha+4)}x^4+\cdots\right]$$

$$y_1=y(x,0)=1-\frac{1}{16}x^4+\cdots$$

$$y_2=\frac{\partial y}{\partial x}\Big|_{x=0}=x^{\alpha}\ln x\left[\alpha-\frac{\alpha}{(\alpha+2)(\alpha-2)}x^2+\cdots\right]$$
$$+x^4\left[1-\left(\frac{\alpha^2-4-2\alpha^2}{(\alpha^2-4)^2}\right)x^2+\cdots\right]_{\alpha=0}$$

$$y=Ay_1+By_2$$

انتهت المحاضرة